IRREGULAR SETS FOR PIECEWISE MONOTONIC MAPS

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ABSTRACT. For any transitive piecewise monotonic map for which the set of periodic measures is dense in the set of ergodic invariant measures (such as monotonic mod one transformations and piecewise monotonic maps with two monotonic pieces), we show that the set of points for which the Birkhoff average of a continuous function does not exist (called the irregular set) is either empty or has full topological entropy. This generalizes Thompson's theorem for irregular sets of β-transformations, and reduces a complete description of irregular sets of transitive piecewise monotonic maps to Hofbauer-Raith problem on the density of periodic measures.

1. INTRODUCTION

Let X be a compact metric space and $T : X \to X$ a Borel measurable map. Let $\mathcal{C}(X)$ be the set of continuous functions on X. The *irregular set* $E(\varphi)$ of $\varphi \in \mathcal{C}(X)$ is given by

$$E(\varphi) = \left\{ x \in X \mid \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(x)) \text{ does not exist} \right\}.$$

It is also called the set of *non-typical points* ([3]) or *divergence points* ([9]), and (the forward orbit of) a point in $\bigcup_{\varphi\in\mathcal{C}(X)} E(\varphi)$ is said to have *historic behavior* ([28, 30]).

Although every irregular set is a μ-zero measure set for any invariant measure μ due to Birkhoff's ergodic theorem, the set is known to be remarkably large for abundant dynamical systems. Pesin and Pitskel [26] obtained the first result for the largeness of irregular sets from thermodynamic viewpoint. In their paper, they showed that every irregular set for the full shift is either empty or has full topological entropy, that is,

$$E(\varphi) = \emptyset \quad \text{or} \quad h_{\mathrm{top}}(T, E(\varphi)) = h_{\mathrm{top}}(T, X)$$

(they also showed that $E(\varphi)$ has full Hausdorff dimension if and only if $E(\varphi) \neq \emptyset$). Here $h_{\mathrm{top}}(T, Z)$ is the (Bowen's Hausdorff) topological entropy for a continuous map T on a (not necessarily compact) Borel set Z given in [7] (see Subsection 2.1 for precise definition; refer to [15] for relation between entropies for a non-compact). It is also known that $E(\varphi) \neq \emptyset$ if and only if $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$ for some ergodic invariant probability measures μ_1, μ_2 (refer to [31, Lemma 1.6]). Pesin-Pitskel's thermodynamic dichotomy for irregular sets was extended to topologically mixing subshifts of finite type in [3] (together with the detailed study of the set of points at

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which Lyapunov exponent or local entropy fail to exist), to continuous maps with specification property in [9] (see also [31]), and to continuous maps with almost specification property (including all β-transformations) by Thompson [32]. We also note that Färm [11] showed a stronger property (called large intersection property) than the full Hausdorff dimension for irregular sets of β-transformations for a large class of β's (this restriction on β was later removed by in [13]), independently of the Thompson's work. See also [1, 2, 5, 6, 12, 23, 28, 30] and references therein for the study of irregular sets from other viewpoints.

The aim of this paper is to extend the thermodynamic dichotomy to transitive piecewise monotonic maps for which the set of periodic measures is dense in the set of ergodic invariant measures. We emphasize that we do not (explicitly) assume any specification-like property on T, being in contrast to all the previous works (see also the remark above Proposition 3.1). Furthermore, we will see that the transitivity seems to be intractable (Remark 3.3; see also Remark 3.2 for necessity of the density of periodic measures) and that the density of periodic measures is shown to hold for abundant classes of transitive piecewise monotonic maps by several authors while no transitive piecewise monotonic map without the density of periodic measures is presently known at now (see Subsection 1.2 for detail). Hence we hope that our result would be a nice step to a complete description of irregular sets of piecewise monotonic maps.

1.1. **Main results.** Let $T : X \to X$ be a Borel measurable map on a metric space X. We say that T is *transitive* if there exists a point $x \in X$ whose forward orbit $\{T^n(x) \mid n \geq 0\}$ is dense in X. We denote by $\mathcal{M}_T^{\mathrm{erg}}(X)$ the set of ergodic T-invariant Borel probability measures on X, endowed with the weak topology. A probability measure μ is called a periodic measure if there is a periodic point p of period n such that $\mu = \big(\sum_{j=0}^{n-1} \delta_{T^j(p)}\big)/n$, where δ_y is the Dirac measure at $y \in X$, and we let $\mathcal{M}_T^{\mathrm{per}}(X)$ be the set of periodic measures on X. Note that $\mathcal{M}_T^{\mathrm{per}}(X) \subset \mathcal{M}_T^{\mathrm{erg}}(X)$. Finally, a Borel measurable map $T : [0,1] \to [0,1]$ on the interval $[0,1]$ is said to be a *piecewise monotonic map* if there are disjoint open intervals $I_1, \ldots, I_k$ such that $[0,1] \setminus \bigcup_{j=1}^k I_j$ is a finite set and $T|_{I_j}$ is monotonic and continuous for each $1 \leq j \leq k$. We denote by $h_{\mathrm{top}}(T, Z)$ the topological entropy for a piecewise monotonic map T on a Borel set Z, which was introduced by Hofbauer [20] as a generalization of the Bowen's Hausdorff topological entropy for continuous maps (see Subsection 2.1 for precise definition).

Our main theorem is as follows:

Theorem 1. *Let $T : [0,1] \to [0,1]$ be a transitive piecewise monotonic map. Suppose that $\mathcal{M}_T^{\mathrm{per}}([0,1])$ is dense in $\mathcal{M}_T^{\mathrm{erg}}([0,1])$. Then, for each $\varphi \in \mathcal{C}([0,1])$, either $E(\varphi) = \emptyset$ or $h_{\mathrm{top}}(T, E(\varphi)) = h_{\mathrm{top}}(T, [0,1])$.*

Theorem 1 is a generalization of Thompson's theorem [32, Theorem 5.1] for irregular sets of β-transformations. We will reduce Theorem 1 to the following analogous result on coding spaces (see Subsection 2.3 for definitions).

Theorem 2. *Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map, Σ_T^+ the coding space of T and $\sigma\colon \Sigma_T^+ \to \Sigma_T^+$ the left shift operator. Suppose that $\mathcal{M}_\sigma^{\mathrm{per}}(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_T^+)$. Then for each $\varphi \in \mathcal{C}(\Sigma_T^+)$, either $E(\varphi) = \emptyset$ or $h_{\mathrm{top}}(\sigma, E(\varphi)) = h_{\mathrm{top}}(\sigma, \Sigma_T^+)$.*

1.2. **Applications.** We will see in Remark 3.3 that one can easily construct *non-transitive* piecewise monotonic maps T and continuous maps φ for which the dichotomy "$E(\varphi) = \emptyset$ or $h_{\mathrm{top}}(T, E(\varphi)) = h_{\mathrm{top}}(T, [0, 1])$" in Theorem 1 do not hold, so the transitivity condition is indispensable for our purpose. On the other hand, the property that $\mathcal{M}_T^{\mathrm{per}}([0, 1])$ is dense in $\mathcal{M}_T^{\mathrm{erg}}([0, 1])$ has been intensively studied by many authors independently from irregular sets (e.g. [4, 18, 19, 21]) and shown to hold for a large class of piecewise monotonic maps. We recall that Hofbauer and Raith [21, 27] proposed a problem asking whether $\mathcal{M}_T^{\mathrm{per}}([0, 1])$ is dense in $\mathcal{M}_T^{\mathrm{erg}}([0, 1])$ for any transitive piecewise monotonic maps with positive topological entropy. The Hofbauer-Raith problem has a positive answer in the following three important cases. In these cases, we can apply Theorems 1 and 2 if the map T is transitive:

- The map T is a *continuous* map with positive topological entropy. It follows from [4, Corollary 10.5] that $\mathcal{M}_T^{\mathrm{per}}([0, 1])$ is dense in $\mathcal{M}_T^{\mathrm{erg}}([0, 1])$ in this case. The result was motivated by the density of periodic measures for continuous maps with the specification property ([10]).
- The map T is a *monotonic mod one transformation* (that is, there exists a strictly increasing and continuous function $f : [0, 1] \to \mathbb{R}$ such that $T(x) = f(x) \mod 1$; this is also called a *Lorenz map*) with positive topological entropy. The density of periodic measures was proven in [18, Theorem 2]. A special case of this class is a *linear mod one transformation*, which was first introduced by Parry ([25]) and defined by

$$T(x) = \begin{cases} \beta x + \alpha \mod 1 & (x \in [0, 1)) \\ \lim_{y \to 1-0} (\beta y + \alpha \mod 1) & (x = 1) \end{cases}$$

  with $\beta > 1$ and $0 \leq \alpha < 1$. When $\alpha = 0$, the map T is called the *β-transformation* (its irregular sets were investigated in [32]). Another special case of monotonic mod one transformations appears in the Poincaré maps of geometric Lorentz flows (refer to [22] in which irregular sets for geometric Lorentz flows were shown to be residual).
- The map T has *two* intervals of monotonicity with positive topological entropy. That is, there is a point $0 < a < 1$ such that both $T|_{(0,a)}$ and $T|_{(a,1)}$ are monotonic and continuous. The density of periodic measures in this case was shown in [21, Theorem 2].

In particular, we can and do apply Theorems 1 and 2 to all transitive monotonic mod one transformations and all transitive piecewise monotonic maps with two monotonic pieces. To the best of our knowledge, this is the first result for Pesin-Pitskel type dichotomy for irregular sets of these transformations.

2. Preliminaries

2.1. **Topological entropy for non-compact sets.** In this subsection, we recall the definition of the topological entropy for non-compact sets. Let X be a compact metric space with a metric d. We endow it with the Borel σ-field. Let $T : X \to X$ be a measurable map.

Firstly, we briefly recall the definition of the (Bowen's Hausdorff) topological entropy $h_{\mathrm{top}}(T, Z)$ for any continuous map T and Borel subset $Z \subset X$. For $n \geq 1$, $x \in X$ and $\epsilon > 0$, we let $B_n(x, \epsilon)$ be the ϵ-ball of center x with respect to the

n-th Bowen-Dinaburg metric d_n given by $d_n(x,y) = \max\{d(T^j(x), T^j(y)) \mid 0 \leq j \leq n-1\}$. For $s \in \mathbb{R}$, $L \in \mathbb{N}$ and $\epsilon > 0$, we set

$$M(Z, s, L, \epsilon) = \inf_{\Gamma} \left\{ \sum_{B_{n_i}(x_i,\epsilon)\in\Gamma} e^{-sn_i} \right\}$$

where the infimum is taken over all $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ being a finite or countable cover of Z such that $x_i \in X$ and $n_i \geq L$ for all i, with the convention $M(\emptyset, s, L, \epsilon) = 0$. Since the quantity $M(Z, s, L, \epsilon)$ does not decrease with L, we can define $m(Z, s, \epsilon)$ given by

$$m(Z, s, \epsilon) = \lim_{L\to\infty} M(Z, s, L, \epsilon).$$

Define

$$h_{\mathrm{top}}(T, Z, \epsilon) = \inf\{s \in \mathbb{R} \mid M(Z, s, \epsilon) = 0\} = \sup\{s \in \mathbb{R} \mid M(Z, s, \epsilon) = \infty\},$$

whose existence is easily seen by the standard argument. Finally we define

$$h_{\mathrm{top}}(T, Z) = \lim_{\epsilon\to 0} h_{\mathrm{top}}(T, Z, \epsilon)$$

and call it the *topological entropy* of Z.

It is easy to see that $h_{\mathrm{top}}(T, Z_1) \leq h_{\mathrm{top}}(T, Z_2)$ if $Z_1 \subset Z_2 \subset Y$. Moreover, $h_{\mathrm{top}}(T, X)$ coincides with the usual topological entropy of $T : X \to X$ (we here mean by the usual topological entropy the topological entropy in the sense of Adler-Konheim-McAndrew or Bowen-Dinaburg; see e.g. [15]). In particular, by the classical variational principle (for the Adler-Konheim-McAndrew topological entropy in [14]; see also [33, Corollary 8.6.1.(i)]), we have

$$h_{\mathrm{top}}(T, X) = \sup_{\mu\in\mathcal{M}_T^{\mathrm{erg}}(X)} h(\mu), \tag{2.1}$$

where $h(\mu)$ denotes the *metric entropy* of μ.

Next we recall the definition of the (Hofbauer's) topological entropy for a piecewise monotonic map T. Let $\mathcal{J}$ be a partition of $[0,1]$ consisting of monotonicity open intervals $I_1, \ldots, I_k$ of T (i.e. $I_1, \ldots, I_k$ are disjoint, $[0,1] \setminus \bigcup_{j=1}^k I_j$ is a finite set and $T|_{I_j}$ is monotonic and continuous for each $1 \leq j \leq k$). Set $\mathcal{J}_\ell = \bigvee_{i=0}^{\ell-1} T^{-i}\mathcal{J}$ for $\ell \geq 1$ and $\mathcal{K}_L = \bigcup_{\ell=L}^\infty \mathcal{J}_\ell$ for $L \geq 1$. For $s \geq 0$ and $Z \subset [0,1]$, we define

$$\widetilde{m}(Z, s) = \lim_{L\to\infty} \widetilde{M}(Z, s, L),$$

where

$$\widetilde{M}(Z, s, L) = \inf_{\Gamma} \sum_{B\in\Gamma} e^{-sn(B)},$$

where the infimum is taken over the set of all finite or countable subsets Γ of $\mathcal{K}_L$ with $Z \subset \bigcup_{B\in\Gamma} B$ and $n(B)$ is the maximal integer ℓ such that $B \in \mathcal{J}_\ell$. It is shown in [20, Lemma 2] that for each subset Z of $[0,1]$, there is $s_0 \geq 0$ independently of the choice of $\mathcal{J}$ such that $\widetilde{m}(Z, s) = \infty$ for all $s < s_0$ and $\widetilde{m}(Z, s) = 0$ for all $s > s_0$. Furthermore, this value is proven to coincide with the Bowen's Hausdorff entropy of Z whenever Z is a subset of the maximal continuity set X_T,

$$X_T = [0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}(\{i_0, \ldots, i_k\}), \tag{2.2}$$

where $i_0, \dots, i_k$ are the endpoints of $I_1, \dots, I_k$ ([20, Lemma 3]; note that X_T is invariant and $T : X_T \to X_T$ is continuous). Hence, we can consistently use the notation $h_{\mathrm{top}}(T, Z)$ to denote the value s_0. Note also that $\bigcup_{n=0}^{\infty} T^{-n}(\{i_0, \dots, i_k\})$ is only countable, so it follows from the argument above Lemma 1 of [20] that

$$h_{\mathrm{top}}(T, Z \cap X_T) = h_{\mathrm{top}}(T, Z) \tag{2.3}$$

for any subset Z of $[0, 1]$.

2.2. **Symbolic dynamics.** Let D be a countable set. Denote by $D^{\mathbb{N}}$ the one-sided infinite product of D equipped with the product topology of the discrete topology of D. Let σ be the left shift operator of $D^{\mathbb{N}}$ (i.e. $(\sigma(x))_j = x_{j+1}$ for each $j \in \mathbb{N}$ and $x = (x_j)_{j\in\mathbb{N}} \in D^{\mathbb{N}}$). When a subset Σ^+ of $D^{\mathbb{N}}$ is σ-invariant and closed, we call it a *subshift*, and D the *alphabet* of Σ^+. When Σ^+ is of the form

$$\Sigma^+ = \{x \in D^{\mathbb{N}} \mid M_{x_j x_{j+1}} = 1 \text{ for all } j \in \mathbb{N}\}$$

with a matrix $M = (M_{ij})_{(i,j)\in D^2}$ each entry of which is 0 or 1, we call Σ^+ a *Markov shift*. When we emphasize the dependence of Σ^+ on M, it is denoted by Σ^+_M, and M is called the *adjacency matrix* of Σ^+_M.

For a subshift Σ^+ on an alphabet D, let $[x] = \{y \in \Sigma^+ \mid (y_1, \dots, y_n) = x\}$ for each $x \in D^n$, $n \geq 1$, and set $\mathcal{L}(\Sigma^+) = \big\{x \in \bigcup_{n\geq 1} D^n \mid [x] \neq \emptyset\big\}$. We say that a subshift Σ^+ on a finite alphabet satisfies the *specification property* if there is an integer $L > 0$ such that for any $x, y \in \mathcal{L}(\Sigma^+)$, one can find $z \in \mathcal{L}(\Sigma^+)$ with $|z| \leq L$ such that $xzy \in \mathcal{L}(\Sigma^+)$, where $|z|$ is the integer n such that $z \in D^n$. (Notice that Bowen [8] originally called the specification property the above condition with $|z| = L$ instead of $|z| \leq L$, which seems to be more standard among dynamicists, but the difference makes no influence on this paper and we permit us to adopt our definition for simplicity.) The following is elementary but important in the proof of Theorem 2.

Lemma 2.1. *Let Σ^+_M be a transitive Markov shift with an adjacency matrix M on a finite alphabet D. Then Σ^+_M satisfies the specification property.*

Proof. Since Σ^+_M is transitive, for any $(i, j) \in D^2$, there exists an integer $L(i, j) > 0$ such that $M_{ij}^{L(i,j)} > 0$. We set $L = \max_{(i,j)\in D^2} L(i, j)$. Let $x, y \in \mathcal{L}(\Sigma^+_M)$ and denote by n the length of x. Since $M_{x_n y_1}^{L(x_n,y_1)} > 0$, we can find $z_1 \cdots z_\ell \in \mathcal{L}(\Sigma^+_M)$ such that $\ell = L(x_n, y_1) - 1 \leq L$ and $M_{x_n z_1} = M_{z_\ell y_1} = 1$, which imply that $xzy \in \mathcal{L}(\Sigma^+_M)$. □

2.3. **Markov diagram for piecewise monotonic maps.** Fix a transitive piecewise monotonic map $T : [0, 1] \to [0, 1]$ and let X_T be the invariant set given in (2.2). Define the coding map $\mathcal{I} : X_T \to \{1, \dots, k\}^{\mathbb{N}}$ of T by

$$(\mathcal{I}(x))_j = \ell \quad \text{if } T^{j-1}(x) \in I_\ell.$$

We note that $\mathcal{I}$ is well-defined and injective since T is transitive. Denote the closure of $\mathcal{I}(X_T)$ by Σ^+_T and call it the *coding space* of T. Observe that Σ^+_T is a compact σ-invariant set. Again by the transitivity of T, we can easily see that Σ^+_T is transitive. In what follows we will construct Hofbauer's Markov diagram, which is a countable oriented graph with subsets of Σ^+_T as vertices.

Let $D \subset \Sigma^+_T$ be a closed subset with $D \subset [i]$ for some $1 \leq i \leq k$. We say that a non-empty closed subset $C \subset \Sigma^+_T$ is a *successor* of D if $C = [j] \cap \sigma(D)$ for some $1 \le j \le k$. Now we define a set $\mathcal{D}$ of vertices by induction. First, we set $\mathcal{D}_0 = \{[1], \dots, [k]\}$. If $\mathcal{D}_n$ is defined for $n \ge 0$, then we define $\mathcal{D}_{n+1}$ by

$$\mathcal{D}_{n+1} = \left\{ C \subset \Sigma_T^+ \mid \text{there exists } D \in \mathcal{D}_n \text{ such that } C \text{ is a successor of } D \right\}.$$

We note that $\mathcal{D}_n$ is a finite set for each n since the number of successors of any closed subset of Σ_T^+ is at most k by the definition. Finally, we set

$$\mathcal{D} = \bigcup_{n \ge 0} \mathcal{D}_n.$$

To get the oriented graph, which we call *Hofbauer's Markov diagram*, we insert an arrow from every $D \in \mathcal{D}$ to all of the successors of D. We write $D \to C$ to denote that C is a successor of D. We define a matrix $M(\mathcal{D}) = (M_{DC})_{(D,C) \in \mathcal{D}^2}$ by

$$M_{DC} = \begin{cases} 1 & (D \to C), \\ 0 & (\text{otherwise}). \end{cases}$$

Then $\Sigma_{M(\mathcal{D})}^+ = \{(D_i)_{i \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}} \mid D_i \to D_{i+1} \text{ for all } i \in \mathbb{N}\}$ is a Markov shift with a countable alphabet $\mathcal{D}$ and an adjacency matrix $M(\mathcal{D})$. We define $\Psi \colon \Sigma_{M(\mathcal{D})}^+ \to \{1, \dots, k\}^{\mathbb{N}}$ by

$$\Psi((D_i)_{i \in \mathbb{N}}) = (x_i)_{i \in \mathbb{N}} \quad \text{for } (D_i)_{i \in \mathbb{N}} \in \Sigma_{M(\mathcal{D})}^+,$$

where $1 \le x_i \le k$ is the unique integer such that $D_i \subset [x_i]$ holds for each $i \in \mathbb{N}$. Then it is clear that Ψ is continuous, countable-to-one, and satisfies $\Psi \circ \sigma = \sigma \circ \Psi$. We remark that $\Sigma_{M(\mathcal{D})}^+$ is not transitive in general although Σ_T^+ is transitive. We use the following two theorems shown by Hofbauer.

Theorem 2.2. ([17, Theorem 11]) *Suppose that $h_{\mathrm{top}}(\sigma, \Sigma_T^+) > 0$. Then we can find a subset $\mathcal{C}_0 \subset \mathcal{D}$ such that $\Sigma_{M(\mathcal{C}_0)}^+$ is transitive and $\Psi(\Sigma_{M(\mathcal{C}_0)}^+) = \Sigma_T^+$. Here $M(\mathcal{C}) = (M_{DC}(\mathcal{C}))_{(D,C) \in \mathcal{C}^2}$ denotes the submatrix of $M(\mathcal{D})$ for $\mathcal{C} \subset \mathcal{D}$.*

Theorem 2.3. ([16, Theorems 1 and 2]) *Suppose that $h_{\mathrm{top}}(\sigma, \Sigma_T^+) > 0$ and let $\mathcal{C}_0 \subset \mathcal{D}$ be as in Theorem 2.2. Then there is a unique measure maximizing entropy both on $\Sigma_{M(\mathcal{C}_0)}^+$ and Σ_T^+, i.e. there is a unique probability measure $\widetilde{m}$ on $\Sigma_{M(\mathcal{C}_0)}^+$ and a unique probability measure m on Σ_T^+ such that $h(\widetilde{m}) = h(m) = h_{\mathrm{top}}(\sigma, \Sigma_T^+)$. Furthermore, it holds that $m = \widetilde{m} \circ \Psi^{-1}$.*

To prove Theorem 2, we also use the following lemma:

Lemma 2.4. *Suppose that $h_{\mathrm{top}}(\sigma, \Sigma_T^+) > 0$. Let $\mathcal{C}_0$ be as in Theorem 2.2 and let $\mathcal{F}_1$, $\mathcal{F}_2$ be finite subsets of $\mathcal{C}_0$. Then, one can find a subset $\mathcal{F}$ of $\mathcal{C}_0$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}$ and $\Psi(\Sigma_{M(\mathcal{F})}^+)$ satisfies the specification property.*

Proof. It is straightforward to see that, by the transitivity of $\Sigma_{M(\mathcal{C}_0)}^+$, there is a finite subset $\mathcal{F}$ of $\mathcal{C}_0$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}$ and $\Sigma_{M(\mathcal{F})}^+$ is a transitive Markov shift (on a finite alphabet $\mathcal{F}$). Hence it follows from Lemma 2.1 that $\Sigma_{M(\mathcal{F})}^+$ satisfies the specification property. It is a well-known fact that the factor of a system with the specification property has the specification property (cf. [10, (21.4) Proposition (c)]). By the property of Ψ noted above, we have that $\Psi \colon \Sigma_{M(\mathcal{F})}^+ \to \Psi(\Sigma_{M(\mathcal{F})}^+)$ is a factor map, which completes the proof of the lemma. □

3. The proofs of main theorems

In this section, we give the proofs of Theorems 1 and 2. Let Σ_T^+ be as in Theorem 2 and suppose that $\mathcal{M}_\sigma^{\mathrm{per}}(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_T^+)$. Let $\varphi \in \mathcal{C}(\Sigma_T^+)$ and assume that $E(\varphi) \neq \emptyset$. We will show that $h_{\mathrm{top}}(\sigma, E(\varphi)) = h_{\mathrm{top}}(\sigma, \Sigma_T^+)$. Since $h_{\mathrm{top}}(\sigma, \Sigma_T^+) = 0$ immediately implies $h_{\mathrm{top}}(\sigma, E(\varphi)) = 0$ (and thus $h_{\mathrm{top}}(\sigma, E(\varphi)) = h_{\mathrm{top}}(\sigma, \Sigma_T^+)$), we may assume that $h_{\mathrm{top}}(\sigma, \Sigma_T^+) > 0$. By [31, Lemma 1.6], the assumption that $E(\varphi) \neq \emptyset$ implies

$$\inf_{\mu \in \mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_T^+)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_T^+)} \int \varphi d\mu. \tag{3.1}$$

It was proven in the previous works [3, 9, 32] that if Σ_T^+ satisfies (a weaker form of) the specification property, then $E(\varphi) \neq \emptyset$ implies $h_{\mathrm{top}}(\sigma, E(\varphi)) = h_{\mathrm{top}}(\sigma, \Sigma_T^+)$. However, in our setting, it seldom happens that Σ_T^+ has the specification property and it is unclear if there is a specification-like property satisfied by all Σ_T^+, which makes the proof of Theorem 2 difficult. To overcome the difficulty, we employed a strategy to find a subset of Σ_T^+ which satisfies the specification property with *almost* full topological entropy:

Proposition 3.1. *For any $\epsilon > 0$, there is a compact σ-invariant set $\Sigma_\epsilon^+ \subset \Sigma_T^+$ such that the following holds.*

(I) Σ_ϵ^+ *satisfies the specification property.*

(II) $\mathcal{M}_\sigma(\Sigma_\epsilon^+)$ *is nontrivial with respect to φ:*

$$\inf_{\mu \in \mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_\epsilon^+)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_\epsilon^+)} \int \varphi d\mu.$$

(III) $h_{\mathrm{top}}(\sigma, \Sigma_\epsilon^+) \geq h_{\mathrm{top}}(\sigma, \Sigma_T^+) - \epsilon.$

Proof. Let m be the unique measure maximizing entropy on Σ_T^+ in Theorem 2.3. Then (3.1) implies that there is an ergodic invariant probability measure μ on Σ_T^+ such that $\int \varphi d\mu \neq \int \varphi dm$. Therefore, by the assumption that $\mathcal{M}_\sigma^{\mathrm{per}}(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_T^+)$, one can find $\mu_{\mathrm{per}} \in \mathcal{M}_\sigma^{\mathrm{per}}(\Sigma_T^+)$ such that $\int \varphi d\mu_{\mathrm{per}} \neq \int \varphi dm$. Let α be a positive number such that $|\int \varphi d\mu_{\mathrm{per}} - \int \varphi dm| > 2\alpha$ holds. Choose an open neighborhood $\mathcal{U}$ of m in $\mathcal{M}_\sigma(\Sigma_T^+)$ such that $|\int \varphi dm - \int \varphi d\mu| \leq \alpha$ whenever $\mu \in \mathcal{U}$. Let $\mathcal{C}_0 \subset \mathcal{D}$ be as in Theorem 2.2 and $\widetilde{m}$ the unique measure maximizing entropy on $\Sigma_{M(\mathcal{C}_0)}^+$ such that $m = \widetilde{m} \circ \Psi^{-1}$ in Theorem 2.3. Since Ψ is continuous, we can find an open neighborhood $\mathcal{V}$ of $\widetilde{m}$ such that for any invariant measure $\widetilde{\mu} \in \mathcal{V}$, we have $\widetilde{\mu} \circ \Psi^{-1} \in \mathcal{U}$.

Since $\Sigma_{M(\mathcal{C}_0)}^+$ is transitive, it follows from the entropy-approachability theorem for transitive Markov shifts [29, Main Theorem] that for any $\epsilon > 0$, we can find a finite subset $\mathcal{F}_1 \subset \mathcal{C}_0$ and an ergodic measure $\widetilde{\nu} \in \mathcal{V} \cap \mathcal{M}_\sigma^{\mathrm{erg}}(\Sigma_{M(\mathcal{F}_1)}^+)$ such that $h(\widetilde{\nu}) \geq h(\widetilde{m}) - \epsilon = h(m) - \epsilon$ holds. We set $\nu = \widetilde{\nu} \circ \Psi^{-1}$. Clearly, $\nu \in \mathcal{U} \cap \mathcal{M}_\sigma^{\mathrm{erg}}(\Psi(\Sigma_{M(\mathcal{F}_1)}^+))$. Since $\Psi \colon \Sigma_{M(\mathcal{F}_1)}^+ \to \Psi(\Sigma_{M(\mathcal{F}_1)}^+)$ is a countable-to-one factor map, we can see that $h(\widetilde{\nu}) = h(\nu)$ (cf. [24, Theorem 2.1]). Hence we have $h(\nu) \geq h_{\mathrm{top}}(\sigma, \Sigma_T^+) - \epsilon$.

Let $x \in \Sigma_T^+$ be a periodic point in the support of μ_{per}. Then, by $\Psi(\Sigma_{M(\mathcal{C}_0)}^+) = \Sigma_T^+$ and [17, Theorem 8], there are finitely many vertices $C_1, \ldots, C_n \in \mathcal{C}_0$ such that $\Psi(C_1 \cdots C_n C_1 \cdots C_n \cdots) = x$. We set $\mathcal{F}_2 = \{C_1, \ldots, C_n\}$. Since both $\mathcal{F}_1$ and $\mathcal{F}_2$

are finite subsets of $\mathcal{C}_0$, it follows from Lemma 2.4 that one can find a subset $\mathcal{F}$ of $\mathcal{C}_0$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}$ and $\Psi(\Sigma^+_{M(\mathcal{F})})$ satisfies the specification property.

Set $\Sigma^+_\epsilon = \Psi(\Sigma^+_{M(\mathcal{F})})$. We have already shown (I). Note that $\nu, \mu_{\mathrm{per}} \in \mathcal{M}^{\mathrm{erg}}_\sigma(\Sigma^+_\epsilon)$. Since $\nu \in \mathcal{U}$, we have $|\int \varphi d\nu - \int \varphi d\mu_{\mathrm{per}}| \geq |\int \varphi dm - \int \varphi d\mu_{\mathrm{per}}| - |\int \varphi d\nu - \int \varphi dm| \geq 2\alpha - \alpha > 0$, which implies (II). Finally by the variational principle in (2.1),

$$h_{\mathrm{top}}(\sigma, \Sigma^+_\epsilon) \geq h(\nu) \geq h_{\mathrm{top}}(\sigma, \Sigma^+_T) - \epsilon,$$

which implies (III) and completes the proof of Proposition 3.1. □

Remark 3.2. The first paragraph in the proof of Proposition 3.1 is the only place where the density of periodic measures is used. It is natural to ask if we can remove the condition by directly applying [29, Main Theorem] to a lift of the contrasting measure μ (i.e. $\mu \in \mathcal{M}^{\mathrm{erg}}_\sigma(\Sigma^+_T)$ satisfying $\int \varphi d\mu \neq \int \varphi dm$). However, it is far from obvious that the measure μ can be lifted as a measure on $\Sigma^+_{M(\mathcal{C}_0)}$ by Ψ because the pushforward map $\widetilde{\mu} \mapsto \widetilde{\mu} \circ \Psi^{-1}$ from $\mathcal{M}^{\mathrm{erg}}_\sigma(\Sigma^+_{M(\mathcal{C}_0)})$ to $\mathcal{M}^{\mathrm{erg}}_\sigma(\Sigma^+_T)$ is not surjective in general (although $\Psi : \Sigma^+_{M(\mathcal{C}_0)} \to \Sigma^+_T$ is surjective; notice that the countable Markov shift $\Sigma^+_{M(\mathcal{C}_0)}$ is not compact in general). This is exactly the reason why we approximated μ by μ_{per} in the proof of Proposition 3.1.

Proof of Theorem 2. Fix $\epsilon > 0$ and let Σ^+_ϵ be the compact σ-invariant subset of Σ^+_T given in Proposition 3.1. Then, it follows from [9, Theorem 3.1] that (I) and (II) of Proposition 3.1 imply the full topological entropy on $E(\varphi|_{\Sigma^+_\epsilon})$:

$$h_{\mathrm{top}}(\sigma, E(\varphi|_{\Sigma^+_\epsilon})) = h_{\mathrm{top}}(\sigma, \Sigma^+_\epsilon).$$

So, (III) leads to that

$$h_{\mathrm{top}}(\sigma, \Sigma^+_T) \geq h_{\mathrm{top}}(\sigma, E(\varphi)) \geq h_{\mathrm{top}}(\sigma, E(\varphi|_{\Sigma^+_\epsilon})) \geq h_{\mathrm{top}}(\sigma, \Sigma^+_T) - \epsilon.$$

Since ϵ is arbitrary, we get $h_{\mathrm{top}}(\sigma, E(\varphi)) = h_{\mathrm{top}}(\sigma, \Sigma^+_T)$, that is, the irregular set $E(\varphi)$ has full topological entropy. This completes the proof. □

Proof of Theorem 1. Let $(I_j)_{j=1}^k$ be the monotonicity intervals of T, and $i_0, \ldots, i_k$ the endpoints of $I_1, \ldots, I_k$ (see Subsection 2.1). Let φ be a continuous function on $[0, 1]$ such that $E(\varphi) \neq \emptyset$. Recall that $\mathcal{M}^{\mathrm{per}}_T([0, 1])$ is supposed to be dense in $\mathcal{M}^{\mathrm{erg}}_T([0, 1])$. Thus, it follows from [34, Theorem A] that $\mathcal{M}^{\mathrm{per}}_\sigma(\Sigma^+_T)$ is also dense in $\mathcal{M}^{\mathrm{erg}}_\sigma(\Sigma^+_T)$. Recall that X_T is given in (2.2).

We first show that $E(\varphi) \cap X_T \neq \emptyset$. Take a point $x \in E(\varphi)$. Arguing by contradiction, we assume that $T^n(x) \notin X_T$ for all $n \geq 0$. Then, $x \notin X_T$, and thus $T^m(x) = i_j$ with some $m \geq 0$ and $0 \leq j \leq k$. Since $T^m(x) \notin X_T$ by assumption, we can find some $m' \geq 0$ and $0 \leq j' \leq k$ such that $T^{m'+m}(x) = i_{j'}$. By repeating this argument (at most) k-times, we conclude that each i_j is a periodic point. It contradicts with that $x \in E(\varphi)$ and $T^m(x) = i_j$. Hence there is an integer $n \geq 0$ such that $T^n(x) \in X_T$. Since $E(\varphi)$ is an invariant set, this implies that $E(\varphi) \cap X_T \neq \emptyset$.

Note that the coding map $\mathcal{I}\colon X_T \to \Sigma^+_T$ is injective by the transitivity of T (refer to §6 in [34]). Hence one can see that the set

$$\bigcap_{n \geq 0} \mathrm{cl}(T^{-n}(I_{x_{n+1}})) \tag{3.2}$$

is a unit set for any $(x_n)_{n\in\mathbb{N}} \in \Sigma_T^+$. Here $\mathrm{cl}(A)$ denotes the closure of the set A. Then we define a map $\Phi\colon \Sigma_T^+ \to [0,1]$ by $\Phi((x_n)_{n\in\mathbb{N}}) = y$, where y is the unique element of the unit set in (3.2). Then it is well known that Φ is continuous, surjective and $\Phi \circ \sigma(x) = T \circ \Phi(x)$ for any $x \in \mathcal{I}(X_T)$. Moreover, we have $\Phi(\mathcal{I}(X_T)) = X_T$ and the restricted map $\Phi\colon \mathcal{I}(X_T) \to X_T$ is the inverse map of $\mathcal{I}\colon X_T \to \mathcal{I}(X_T)$. Hence we have $\varphi \circ \Phi \in \mathcal{C}(\Sigma_T^+)$ and

$$E(\varphi \circ \Phi) \cap \mathcal{I}(X_T) = \mathcal{I}(E(\varphi) \cap X_T).$$

This implies that $E(\varphi\circ\Phi) \neq \emptyset$ since $E(\varphi)\cap X_T \neq \emptyset$, and that $h_{\mathrm{top}}(T, E(\varphi)\cap X_T) = h_{\mathrm{top}}(\sigma, E(\varphi \circ \Phi))$ since $\Sigma_T^+ \setminus \mathcal{I}(X_T)$ is countable. Hence, it follows from Theorem 2 that

$$h_{\mathrm{top}}(\sigma, E(\varphi \circ \Phi)) = h_{\mathrm{top}}(\sigma, \Sigma_T^+).$$

Therefore, by virtue of (2.3), we have

$$\begin{aligned} h_{\mathrm{top}}(T, E(\varphi)) &= h_{\mathrm{top}}(T, E(\varphi) \cap X_T) \\ &= h_{\mathrm{top}}(\sigma, \Sigma_T^+) = h_{\mathrm{top}}(T, X_T) = h_{\mathrm{top}}(T, [0,1]), \end{aligned}$$

which completes the proof. □

Remark 3.3. Let $T : [0,1] \to [0,1]$ be a (not necessarily transitive) piecewise monotone map for which $\mathcal{M}_T^{\mathrm{per}}([0,1])$ is dense in $\mathcal{M}_T^{\mathrm{erg}}([0,1])$. Assume that there are mutually disjoint invariant intervals $J_1, \ldots, J_N$ with $N \in \mathbb{N}$ such that $\bigcup_{j=1}^N J_j = [0,1]$ and the restriction of T on J_j is transitive for each $1 \leq j \leq N$. Then for any continuous function $\varphi : [0,1] \to \mathbb{R}$ for which $E(\varphi) \neq \emptyset$, it follows from Theorem 1 and (2.3) together with [31, Lemma 1.6] that

$$\begin{aligned} h_{\mathrm{top}}(T, E(\varphi)) = \sup \Big\{ & h_{\mathrm{top}}(T, J_j) \mid 1 \leq j \leq N \text{ such that} \\ & \inf_{\mu \in \mathcal{M}_T^{\mathrm{erg}}(J_j \cap X_T)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_T^{\mathrm{erg}}(J_j \cap X_T)} \int \varphi d\mu \Big\}. \end{aligned}$$

As a consequence, one can easily construct *non-transitive* piecewise monotonic maps T and continuous functions φ for which the dichotomy "$E(\varphi) = \emptyset$ or $h_{\mathrm{top}}(T, E(\varphi)) = h_{\mathrm{top}}(T, [0,1])$" do not hold (for example, take J_1, J_2 with $h_{\mathrm{top}}(T, J_1) < h_{\mathrm{top}}(T, J_2)$ and φ supported on J_1 with $\inf_{\mu\in\mathcal{M}_T^{\mathrm{erg}}(J_1\cap X_T)} \int \varphi d\mu < \sup_{\mu\in\mathcal{M}_T^{\mathrm{erg}}(J_1\cap X_T)} \int \varphi d\mu$).

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