TWO APPROACHES TO LARGE DEVIATION PRINCIPLES VIA PERIODIC POINTS BEYOND SPECIFICATION

KENICHIRO YAMAMOTO

Abstract. We generalize Kifer's results for the level-2 large deviation principle (LDP) via periodic points beyond the classical specification setting. We develop two complementary approaches. The first establishes the LDP under the periodic almost specification property in the sense of Thompson. The second uses Markov diagrams to prove the LDP for coding spaces of piecewise monotonic interval maps. Both methods apply not only to β-shifts but also to various non-specification systems, including S-gap shifts, the non-intrinsically ergodic examples of Kwietniak-Oprocha-Rams, (α, β)-shifts, and generalized β-shifts.

Keywords: large deviation principle; periodic points; specification property; almost specification; Markov diagram; symbolic dynamics; piecewise monotonic maps; β-shifts; S-gap shifts; non-intrinsically ergodic subshifts; (α, β)-shifts; generalized β-shifts.

1. Introduction

Large deviation theory provides a quantitative description of the exponential rarity of atypical statistical behavior of orbits in dynamical systems. We briefly recall the general notion of a large deviation principle. Let X be a compact metrizable space and $\mathcal{M}(X)$ be a set of Borel probability measures on X with the weak-topology. We say that a sequence of Borel probability measure $\{\Omega_n\}_{n=1}^{\infty}$ on $\mathcal{M}(X)$ satisfies a *level-2 large deviation principle with rate function* $q : \mathcal{M}(X) \to [-\infty, 0]$ if q is upper semicontinuous, for every closed subset $\mathcal{K} \subset \mathcal{M}(X)$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \Omega_n(\mathcal{K}) \ \leq \ \sup_{\mu\in\mathcal{K}} q(\mu),$$

and if for every open subset $\mathcal{G} \subset \mathcal{M}(X)$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \Omega_n(\mathcal{G}) \ \geq \ \sup_{\mu\in\mathcal{G}} q(\mu).$$

For expansive systems satisfying the specification property, level-2 LDPs for Birkhoff averages were established by Takahashi [32] and Young [40].

2020 *Mathematics Subject Classification.* Primary 37A50, 37B10, 37E05; Secondary 37C30, 60F10.

These results show that the fluctuations of orbit averages with respect to the equilibrium measure satisfy an LDP with the thermodynamic rate function

$$q(\mu) = h(\mu) + \int \varphi \, d\mu - P(\varphi),$$

where $h(\mu)$ denotes the metric entropy of the invariant measure μ, and $P(\varphi)$ is the topological pressure of the continuous observable φ. This formula reveals a deep connection between large deviations and the variational principle for pressure.

Shortly afterward, Kifer [19] proved a periodic-orbit formulation of the LDP under the classical specification property introduced by Bowen [3]. This viewpoint shows that the thermodynamic behavior of a system can often be described in terms of its finite periodic structures.

Recently, this periodic-orbit perspective has been broadened to various symbolic and smooth contexts; see Pollicott–Sridharan [26], Gelfert–Wolf [13], Comman-Rivera-Letelier [9], and Takahasi [34]. Nevertheless, beyond the specification setting, the existing general theory remains rather limited. A canonical example is the β-shift introduced by Rényi [27], which fails specification but still exhibits strong statistical regularity. However, even for such systems, including the β-shift, a general large deviation principle via periodic points has not yet been established.

Motivated by these developments, we first work in the symbolic setting, focusing on subshifts over finite alphabets. Building on this line of research, we develop two complementary approaches that extend Kifer's periodic-orbit LDP framework to systems without the specification property. In particular, these approaches are applicable not only to β-shifts but also to a broader class of subshifts without the specification property. The two methods address different classes of systems.

The first approach relies on the periodic almost specification property, a natural weakening of specification introduced by Thompson [33], which still retains sufficient orbit-gluing structure to support thermodynamic analysis. This allows us to treat systems such as β-shifts, topologically mixing S-gap shifts [22], and examples of Kwietniak-Oprocha–Rams [21] that lack intrinsic ergodicity.

The second approach studies the symbolic coding spaces of piecewise monotonic maps via their Markov diagrams. Considering such coding spaces is natural, an idea tracing back to Rényi [27] and Parry's work on β-transformations and later systematized by Hofbauer [15]. The main assumptions—the density of periodic measures and the irreducibility of the diagram—are classical in this setting. Under these hypotheses, the method yields a periodic-orbit LDP, applicable not only to β-shifts but also to their natural generalizations, including (α, β)-shifts and generalized β-shifts—that is, the coding spaces of (α, β)-transformations [24] and generalized β-transformations [14].

To formulate our result, we introduce the following terminology. For a subshift X over a finite alphabet and a continuous potential $\varphi\colon X \to \mathbb{R}$, we say that the pair (X, φ) satisfies a *level-2 large deviation principle via periodic points* if the associated sequence of probability measures

$$\Omega_n^{\varphi} = \frac{1}{\sum_{x \in P_n(X)} e^{S_n\varphi(x)}} \sum_{x \in P_n(X)} e^{S_n\varphi(x)}\, \delta_{\mathcal{E}_n(x)},$$

on $\mathcal{M}(X)$ satisfies the level-2 LDP stated above. Here $P_n(X)$ denotes the set of periodic points of period n, $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(\sigma^k x)$ is the n-th Birkhoff sum, $\mathcal{E}_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\sigma^j x}$ is the empirical measure associated with x, and δ_y is a Dirac measure at y. The measure Ω_n^{φ} thus represents the distribution of periodic points weighted by the potential φ. Note that earlier results mostly established this principle for Hölder potentials, whereas we show that it holds for every continuous potential.

We now state our two main theorems; see Section 2 for the necessary definitions.

Theorem A. (**Almost-specification approach**). Let X be a subshift over a finite alphabet. If X satisfies the periodic almost specification property, then for every continuous potential $\varphi\colon X \to \mathbb{R}$, (X, φ) satisfies a level-2 large deviation principle via periodic points with rate function $q^{\varphi}\colon \mathcal{M}(X) \to [-\infty, 0]$ expressed by

$$q^{\varphi}(\mu) = \begin{cases} h(\mu) + \int \varphi\, d\mu - P(\varphi) & (\mu \text{ is invariant}), \\ -\infty & (\text{otherwise}). \end{cases} \tag{1.1}$$

This applies to β-shifts, topologically mixing S-gap shifts, and examples of Kwietniak-Oprocha–Rams lacking intrinsic ergodicity.

Theorem B. (Markov-diagram approach). Let $T : [0, 1] \to [0, 1]$ be a piecewise monotonic map with positive topological entropy and let X denotes the coding space of T. Suppose that the following conditions hold:

(I) The set of periodic measures on the coding space X (of T) is dense in the set of ergodic measures on X;
(II) The associated Markov diagram of T is irreducible and has a unit loop.

Then for every continuous potential $\varphi\colon X \to \mathbb{R}$, (X, φ) satisfies a level-2 large deviation principle via periodic points with rate function $q^{\varphi}\colon \mathcal{M}(X) \to [-\infty, 0]$ given by (1.1).

This provides a direct approach to periodic-orbit LDPs for the coding spaces of non-Markov piecewise monotonic maps, including β-shifts, (α, β)-shifts, and generalized β-shifts. In particular, this approach applies even when symbolic structure alone does not guarantee gluing properties, as in the case of (α, β)-shifts and generalized β-shifts, thereby establishing level-2 large deviations via periodic points in these systems.

Remark 1.1. Level–2 large deviation principles for Birkhoff averages beyond the specification setting have been established in several works (e.g.

[6, 8, 25, 29]). These results, however, do not imply the periodic–orbit LDP established in this paper, as the two formulations are of a different nature. For instance, a system may satisfy a Birkhoff–average LDP while the periodic–orbit LDP fails (see Remark 4.1).

Most existing results on the periodic–orbit LDP rely on a functional approach and typically assume that the potential admits a unique equilibrium state. In contrast, our topological approach applies not only beyond the specification property but also to arbitrary continuous potentials, possibly allowing multiple equilibrium states.

Remark 1.2. A preliminary version of part of this work appeared in an earlier non-refereed note by the present author [39], where the periodic-orbit LDP was established only for β-shifts. The present paper extends that non-refereed note by developing two general approaches to large deviation principles via periodic points (Theorems A and B) and applying them to a broader class of dynamical systems, including S-gap shifts, (α, β)-shifts, generalized β-shifts, and non-intrinsically ergodic subshifts in the sense of Kwietniak–Oprocha–Rams. These examples illustrate that the present results extend large deviation principles to systems beyond the specification setting.

The remainder of this paper is organized as follows. Section 2 recalls fundamental definitions and known results from symbolic dynamics and the theory of piecewise monotonic maps, including the notions of (periodic) almost specification and the Markov diagram construction. Section 3 provides the proofs of the two main theorems. Subsection 3.1 establishes the level-2 large deviation principle via periodic points for subshifts with the periodic almost specification property, thereby proving Theorem A. Subsection 3.2 develops the Markov-diagram approach and completes the proof of Theorem B for coding spaces of piecewise monotonic maps. Section 4 discusses concrete applications of these results, including β-shifts, S-gap shifts, non-intrinsically ergodic examples of Kwietniak–Oprocha–Rams, and the symbolic coding spaces of (α, β)- and generalized β-transformations.

2. Definitions and known facts

2.1. **Symbolic dynamics.** Let $\mathbb{N}_0$ denote the set of nonnegative integers and $\mathbb{Z}$ the set of all integers. For a finite or countable alphabet A, we write $A^{\mathbb{N}_0}$ and $A^{\mathbb{Z}}$ for the one-sided and two-sided full shifts over A, respectively, each endowed with the product topology. Here the alphabet A is equipped with the discrete topology. We denote by σ the (left) shift map on both spaces. A closed, shift-invariant subset $\Sigma \subset A^{\mathbb{N}_0}$ (resp. $\Sigma \subset A^{\mathbb{Z}}$) is called a one-sided (resp. two-sided) subshift. Since the two-sided case can be handled analogously up to notational modifications, we shall primarily work with one-sided subshifts in what follows. Unless otherwise stated, all definitions, propositions, lemmas, and proofs are given for the one-sided case, and the two-sided versions are completely analogous.

A subshift Σ is called a *Markov shift* (or *topological Markov shift*) if there exists a matrix $M = (M_{a,b})_{(a,b)\in A^2}$ with entries in $\{0,1\}$ such that

$$\Sigma = \{(x_i)_{i\geq 0} \in A^{\mathbb{N}_0} : M_{x_i,x_{i+1}} = 1, \quad \text{for all } i \geq 0\}.$$

In this case, we also say that Σ is a Markov shift with an *adjacency matrix* M.

For a subshift Σ, its *language* is defined by

$$\mathcal{L}(\Sigma) := \{x_0 x_1 \cdots x_{n-1} : (x_i)_i \in \Sigma,\, n \geq 1\},$$

and for each $w \in \mathcal{L}(\Sigma)$, the associated *cylinder* is

$$[w] := \{x \in \Sigma : x_0 \cdots x_{|w|-1} = w\},$$

where $|w|$ denotes the length of w. For $n \geq 1$, we write

$$\mathcal{L}_n(\Sigma) := \{w \in \mathcal{L}(\Sigma) : |w| = n\}$$

for the set of all admissible words of length n. For $u, v \in \mathcal{L}(\Sigma)$, we use the concatenation uv to denote the word obtained by joining them. We say that Σ is *topologically transitive* if for any $u, v \in \mathcal{L}(\Sigma)$ there exists $w \in \mathcal{L}(\Sigma)$ such that $uwv \in \mathcal{L}(\Sigma)$. We say that Σ is *topologically mixing* if for any $u, v \in \mathcal{L}(\Sigma)$, there exists an integer $N \geq 1$ such that for any $n \geq N$, there exists $w \in \mathcal{L}(\Sigma)$ with $|w| = n$ such that $uwv \in \mathcal{L}(\Sigma)$.

We denote by $\mathcal{M}(\Sigma)$ the set of all Borel probability measures on Σ endowed with the weak topology, by $\mathcal{M}_\sigma(\Sigma) \subset \mathcal{M}(\Sigma)$ the set of σ-invariant ones, and by $\mathcal{M}_\sigma^e(\Sigma) \subset \mathcal{M}_\sigma(\Sigma)$ the set of ergodic ones. Throughout this paper, we denote by $h(\mu)$ the metric entropy of $\mu \in \mathcal{M}_\sigma(\Sigma)$.

Hereafter in this subsection, we restrict our attention to subshifts X over a finite alphabet A.

For $n \geq 1$, let

$$P_n(X) := \{x \in X : \sigma^n x = x\}$$

be the set of all periodic points of period n. For $x \in X$, we define its *empirical measure* by

$$\mathcal{E}_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\sigma^j x},$$

where δ_y denotes the *Dirac measure* at y. For convenience, throughout this paper we regard $\mathcal{E}_n$ as a map from X to $\mathcal{M}(X)$, and use the notation

$$\mathcal{E}_n^{-1}(\mathcal{A}) := \{x \in X : \mathcal{E}_n(x) \in \mathcal{A}\}$$

for $n \geq 1$ and $\mathcal{A} \subset \mathcal{M}(X)$. If $x \in P_n(X)$ for some $n \geq 1$, then $\mathcal{E}_n(x)$ is called the *periodic measure*, and we denote by $\mathcal{M}_\sigma^p(X)$ the set of all such measures.

Denote by $C(X)$ the set of real-valued continuous functions on X endowed with the supremum norm $\|\varphi\| := \sup\{|\varphi(x)| : x \in X\}$. Fix a countable dense subset $\{\varphi_n\}_{n\geq 1} \subset C(X)$, and define a compatible metric D on $\mathcal{M}(X)$ by

$$D(\mu,\nu) := \sum_{n=1}^{\infty} \frac{1}{2^{n+1}\|\varphi_n\|} \left| \int \varphi_n \, d\mu - \int \varphi_n \, d\nu \right|.$$

This metric generates the weak topology on $\mathcal{M}(X)$, and $(\mathcal{M}(X), D)$ is compact since X is a subshift over a finite alphabet.

For $\varphi \in C(X)$ and $n \geq 1$, we write

$$S_n\varphi(x) := \sum_{j=0}^{n-1} \varphi(\sigma^j x).$$

The *topological pressure* for $\varphi \in C(X)$ is defined by

$$P(\varphi) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathcal{L}_n(X)} \exp\left(\sup_{x\in[w]} S_n\varphi(x)\right). \tag{2.1}$$

The topological pressure satisfies the usual *variational principle*:

$$P(\varphi) = \sup_{\mu\in\mathcal{M}_\sigma(X)} \left(h(\mu) + \int \varphi \, d\mu\right). \tag{2.2}$$

An invariant measure $\mu_\varphi \in \mathcal{M}_\sigma(X)$ is called an *equilibrium state* for φ if it attains the above supremum. In particular, when $\varphi \equiv 0$, the pressure reduces to the *topological entropy*, $P(0) = h_{\text{top}}(X)$, and the equilibrium states coincide with the *measures of maximal entropy*.

2.2. **Almost specification property.** In this subsection we recall Bowen's classical specification property, which forms the foundation of many results in thermodynamic formalism. We then employ weakened versions of the specification property—specifically, the almost specification property introduced by Thompson and its periodic analogue—which play a central role in the proof of Theorem A. Throughout this subsection, let X be a subshift over a finite alphabet. The classical specification property was introduced by Bowen [3], and we recall the definition below.

Definition 2.1. We say that X satisfies the *specification property* if there exists an integer $M \geq 0$ such that for any finite collection of words $w^{(1)}$, $w^{(2)}, \ldots, w^{(k)} \in \mathcal{L}(X)$, there exist words $v^{(1)}, v^{(2)}, \cdots, v^{(k-1)} \in \mathcal{L}_M(X)$ such that

$$v = w^{(1)}v^{(1)}w^{(2)}v^{(2)}\cdots w^{(k-1)}v^{(k-1)}w^{(k)} \in \mathcal{L}(X).$$

If the cylinder set $[v]$ contains a periodic point with period $|v| + M$, then we say that X satisfies the *periodic specification property*.

We say that a subshift X is *sofic* if X is a topological factor of a Markov shift over a finite alphabet; that is, there exists a continuous surjection $\pi\colon Y \to X$ with $\sigma \circ \pi = \pi \circ \sigma$, where Y is such a Markov shift. It is known that every topologically mixing sofic shift satisfies the periodic specification property (see, for example, [10, Propositions (21.2) and (21.4)]).

Subsequently, Thompson [36] introduced the notion of the almost specification property. This notion generalizes the classical specification property by allowing a vanishing proportion of "mistakes." To quantify such deviations, we use the Hamming distance between finite words: for two words $u = u_0u_1 \cdots u_{n-1}$ and $v = v_0v_1 \cdots v_{n-1}$ of the same length n, let

$$d_H(u, v) := \#\{\, 0 \le i < n : u_i \ne v_i \,\}.$$

We now recall the definition of the almost specification property.

Definition 2.2. We say that X satisfies the *almost specification property* if there exists a mistake function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n)/n \to 0$ as $n \to \infty$ and the following holds: for any finite collection of words $w^{(1)}, w^{(2)}, \dots, w^{(k)} \in \mathcal{L}(X)$, there exist words $v^{(1)}, v^{(2)}, \dots, v^{(k)} \in \mathcal{L}(X)$ such that

- $|w^{(i)}| = |v^{(i)}|$ for every i;
- $d_H(w^{(i)}, v^{(i)}) \le g(|w^{(i)}|)$ for all i; and
- the concatenation $v = v^{(1)}v^{(2)} \cdots v^{(k)}$ belongs to $\mathcal{L}(X)$.

If the cylinder set $[v]$ contains a periodic point with period $|v|$, then we say that X satisfies the *periodic almost specification property.*

It is straightforward to see that the (resp. periodic) specification property implies the (resp. periodic) almost specification property.

In the remainder of this subsection, we recall several known results that will be needed for the proof of Theorem A.

Proposition 2.3. ([25, Proposition 2.1]). Let $\mu \in \mathcal{M}_\sigma^e(X)$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that for every neighborhood $\mathcal{G} \subset \mathcal{M}(X)$ of μ, there exists a subset $\Gamma \subset \mathcal{E}_n^{-1}(\mathcal{G})$ satisfying:

- $\#\Gamma \ge e^{n(h(\mu)-\epsilon)}$;
- for any distinct $(x_i)_i, (y_i)_i \in \Gamma$, $d_H(x_0x_1 \cdots x_{n-1}, y_0y_1 \cdots y_{n-1}) > \delta n$.

The following proposition is derived from a small modification of [38, Theorem 1.2].

Theorem 2.4. ([38, Theorem 1.2]). Suppose that X satisfies the periodic almost specification property. Then for any $\varphi \in C(X)$,

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(X)} e^{S_n\varphi(x)}$$

holds.

Theorem 2.5. ([25, Theorem 2.1]). Suppose that X satisfies the almost specification property. Then for any $\epsilon > 0$, any $\mu \in \mathcal{M}_\sigma(X)$, and any neighborhood $\mathcal{G} \subset \mathcal{M}(X)$ of μ, there exists an ergodic measure $\nu \in \mathcal{G}$ such that $h(\nu) \ge h(\mu) - \epsilon$.

2.3. **Markov diagram for piecewise monotonic maps.** We next recall the Markov diagram construction for piecewise monotonic maps, which provides a symbolic representation of such systems, via countable Markov shifts. We say that $T : [0,1] \to [0,1]$ is a *piecewise monotonic map* if there exist $k \geq 1$ and points

$$0 = c_0 < c_1 < \cdots < c_k < c_{k+1} = 1$$

such that $T|_{(c_j, c_{j+1})}$ is strictly monotonic and continuous for each $0 \leq j \leq k$. For such a map T, we define

$$X_T := \bigcap_{n \geq 0} T^{-n} \left(\bigcup_{j=0}^{k} (c_j, c_{j+1}) \right),$$

and the *coding map* $\mathcal{I} : X_T \to \{0, \ldots, k\}^{\mathbb{N}_0}$ by

$$(\mathcal{I}(x))_n = j \quad \text{iff} \quad T^n(x) \in (c_j, c_{j+1}).$$

We denote by Σ_T the closure of $\mathcal{I}(X_T)$ in $\{0, \ldots, k\}^{\mathbb{N}_0}$, and call Σ_T the *coding space* of T.

Following Hofbauer, we recall the construction of the *Markov diagram*. Let $C \subset \Sigma_T$ be a closed subset contained in the cylinder $[j]$ for some $0 \leq j \leq k$. A non-empty closed subset $D \subset \Sigma_T$ is said to be a *successor* of C if $D = [l] \cap \sigma(C)$ for some $0 \leq l \leq k$. We write $C \to D$ when D is a successor of C.

Define inductively a collection of vertices:

$$\mathcal{D}_0 := \{[0], \ldots, [k]\}, \quad \mathcal{D}_{n+1} := \mathcal{D}_n \cup \{D : C \to D \text{ for some } C \in \mathcal{D}_n\},$$

and let $\mathcal{D}_T := \bigcup_{n \geq 0} \mathcal{D}_n$. Then $\mathcal{D}_T$ is at most countable since $\mathcal{D}_n$ is finite for each $n \geq 0$. The oriented graph $(\mathcal{D}_T, \to)$ is called the *Markov diagram* of T. For notational simplicity, we use the notation $\mathcal{D}$ instead of $\mathcal{D}_T$ if no confusion arises.

For $\mathcal{C} \subset \mathcal{D}$, define the adjacency matrix $M(\mathcal{C}) = (M_{C,D})_{(C,D) \in \mathcal{C}^2}$ by

$$M_{C,D} = \begin{cases} 1 & (C \to D), \\ 0 & (\text{otherwise}). \end{cases}$$

Then

$$\Sigma_{\mathcal{C}} := \{(C_i)_{i \geq 0} \in \mathcal{C}^{\mathbb{N}_0} : C_i \to C_{i+1},\ i \in \mathbb{N}_0\}$$

is a countable-state Markov shift with an adjacency matrix $M(\mathcal{C})$. We say that $\mathcal{C}$ is *irreducible* if for any $C, D \in \mathcal{C}$, there exist an integer $n \geq 1$ and $C_0, \cdots, C_n \in \mathcal{C}$ such that $C_0 = C$, $C_n = D$ and $C_i \to C_{i+1}$ holds for $0 \leq i \leq n-1$. In this case, $\Sigma_{\mathcal{C}}$ is topologically transitive. When, in addition, there exists $C \in \mathcal{C}$ such that $C \to C$, we say that $\mathcal{C}$ is *irreducible with a unit loop*; in that case, $\Sigma_{\mathcal{C}}$ is topologically mixing.

Define a projection

$$\Psi : \Sigma_{\mathcal{D}} \to \{0, \ldots, k\}^{\mathbb{N}_0}, \qquad \Psi((C_i)_i) = (x_i)_i,$$

where x_i is the unique integer such that $C_i \subset [x_i]$. Then it is straightforward to see that Ψ is continuous, countable-to-one, and satisfies $\Psi(\Sigma_{\mathcal{D}}) = \Sigma_T$ and $\sigma \circ \Psi = \Psi \circ \sigma$.

We say that $\mu \in \mathcal{M}^e_\sigma(\Sigma_T)$ is *liftable* if there exists $\bar{\mu} \in \mathcal{M}^e_\sigma(\Sigma_{\mathcal{D}})$ such that $\mu = \bar{\mu} \circ \Psi^{-1}$. Although Ψ is surjective, not every invariant measure is liftable in general. Hofbauer gave a sufficient condition for liftability, which we recall here.

Lemma 2.6. ([15, Lemma 3]). Let $\mu \in \mathcal{M}^e_\sigma(\Sigma_T)$. If μ has positive metric entropy or if μ is a periodic measure, then μ is liftable.

We next recall Takahasi's recent entropy-approximation theorem for topologically transitive countable Markov shifts.

Theorem 2.7. ([33, Main Theorem]). Let Σ be a Markov shift over a countable alphabet. Then for any $\epsilon > 0$, any $\bar{\mu} \in \mathcal{M}_\sigma(\Sigma)$ with $h(\mu) < \infty$ and any neighborhood $\mathcal{G} \subset M(\Sigma)$ of μ, there exist a subshift $\Sigma' \subset \Sigma$ over a finite alphabet and an ergodic measure $\nu \in \mathcal{G}$ such that $h(\nu) \geq h(\mu) - \epsilon$ and $\nu(\Sigma') = 1$.

Finally, we recall the entropy-approximation result from [6] for piecewise monotonic maps under the assumption of Theorem B.

Proposition 2.8. ([6, Proposition 3.1]). Let $T\colon [0,1] \to [0,1]$ be as in Theorem B and let Σ_T denote the coding space of T. Then for any $\epsilon > 0$, any $\mu \in \mathcal{M}_\sigma(\Sigma_T)$ and any neighborhood $\mathcal{G}$ of μ, there exists an ergodic measure $\nu \in \mathcal{G}$ such that $h(\nu) \geq h(\mu) - \epsilon$.

3. Proofs

In this section we prove Theorems A and B. The following known results will be used repeatedly in their proofs.

Theorem 3.1. ([13, Theorem 6]). Let X be a subshift on a finite alphabet. Then for any $\varphi \in C(X)$ and any closed subset $\mathcal{K} \subset \mathcal{M}_\sigma(X)$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(X) \cap \mathcal{E}_n^{-1}(\mathcal{K})} e^{S_n\varphi(x)} \leq \sup_{\mu \in \mathcal{K}} \left(h(\mu) + \int \varphi d\mu \right).$$

3.1. Proof of Theorem A.

In this subsection we present the proof of Theorem A.

Proposition 3.2. Let X be a subshift over a finite alphabet and let $\varphi \in C(X)$. Suppose that X satisfies the periodic almost specification property. Then for any $\mu \in \mathcal{M}^e_\sigma(X)$ and any open neighborhood $\mathcal{G}$ of μ, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(X) \cap \mathcal{E}_n^{-1}(\mathcal{G})} e^{S_n\varphi(x)} \geq h(\mu) + \int \varphi d\mu.$$

Proof. Given any $\epsilon > 0$, choose $\gamma > 0$ sufficiently small so that $\mathbb{B}(\mu, 2\gamma) \subset \mathcal{G}$ and for all $\nu \in \mathbb{B}(\mu, 2\gamma)$,

$$\left| \int \varphi d\mu - \int \varphi d\nu \right| \leq \epsilon,$$

where $\mathbb{B}(\mu, 2\gamma) := \{\nu \in \mathcal{M}(X) : D(\mu, \nu) \leq 2\gamma\}$.

By Proposition 2.3, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for any $n \geq N$, there exists a subset $\Gamma \subset \mathcal{E}^{-1}(\mathbb{B}(\mu, 2\gamma))$ satisfying $\#\Gamma \geq e^{n(h(\mu)-\epsilon)}$ and $d_H(x_0x_1 \cdots x_{n-1}, y_0y_1 \cdots y_{n-1}) > \delta n$ for any distinct $(x_i)_i,\ (y_i)_i \in \Gamma$.

Let $g \colon \mathbb{N} \to \mathbb{N}$ be as in the definition of the periodic almost specification property. By [8, Lemma 2.8], further enlarging N if necessary, for any $n \geq N$, we have $g(n) \leq \delta n$, and for any $x = (x_i)_i,\ y = (y_i)_i \in X$, $D(\mathcal{E}_n(x), \mathcal{E}_n(y)) \leq \gamma$ and $|S_n\varphi(x) - S_n\varphi(y)| \leq \epsilon n$ whenever

$$d_H(x_0x_1 \cdots x_{n-1}, y_0y_1 \cdots y_{n-1}) \leq g(n).$$

By the periodic almost specification property, for each $z = (z_i)_i \in \Gamma$, there exists a periodic point $z^* = (z_i^*)_i \in P_n(X)$ such that

$$d_H(z_0z_1 \cdots z_{n-1}, z_0^*z_1^* \cdots z_{n-1}^*) \leq g(n). \tag{3.1}$$

Then for each $z \in \Gamma$, we have

$$D(\mu, \mathcal{E}_n(z^*)) \leq D(\mu, \mathcal{E}_n(z)) + D(\mathcal{E}_n(z), \mathcal{E}_n(z^*)) \leq 2\gamma. \tag{3.2}$$

and hence $z^* \in \mathcal{E}^{-1}(\mathbb{B}(\mu, 2\gamma)) \subset \mathcal{E}_n^{-1}(\mathcal{G})$.

By the equations (3.1) and (3.2), and the property of Γ, the map

$$z \in \Gamma \mapsto z^* \in P_n(X) \cap \mathcal{E}_n^{-1}(\mathcal{G})$$

is injective and $S_n\varphi(z^*) \geq S_n\varphi(z) - \epsilon n$ holds for any $z \in \Gamma$. Therefore we obtain

$$\begin{aligned} \sum_{x \in P_n(X) \cap \mathcal{E}_n^{-1}(\mathcal{G})} e^{S_n\varphi(x)} &\geq \sum_{z \in \Gamma} e^{S_n\varphi(z^*)} \\ &\geq \sum_{z \in \Gamma} e^{S_n\varphi(z) - \epsilon n} \\ &\geq \sum_{z \in \Gamma} e^{n(\int \varphi d\mu - 2\epsilon)} \\ &= \#\Gamma e^{n(\int \varphi d\mu - 2\epsilon)} \\ &\geq e^{n(h(\mu) + \int \varphi d\mu - 3\epsilon)}. \end{aligned}$$

This completes the proof. □

We now prove Theorem A. Let (X, φ) be as in Theorem A. The upper bound

$$\limsup_{n \to \infty} \frac{1}{n} \log \Omega_n^{\varphi}(\mathcal{K}) \leq \sup_{\mu \in \mathcal{K}} q^{\varphi}(\mu) \qquad (\mathcal{K} \subset \mathcal{M}(X) : \text{ closed})$$

follows directly from Theorems 2.4 and 3.1.

Let $\mathcal{G} \subset \mathcal{M}(X)$ be an open subset and fix $\mu \in \mathcal{G}$. In what follows, we show that

$$\liminf_{n\to\infty} \frac{1}{n} \log \Omega_n^{\varphi}(\mathcal{G}) \geq q^{\varphi}(\mu). \tag{3.3}$$

If μ is not σ-invariant, then $q^{\varphi}(\mu) = -\infty$ so we may, without loss of generality, assume that $\mu \in \mathcal{M}_\sigma(X)$. Given any $\epsilon > 0$, choose a neighborhood $\mathcal{G}^\dagger \subset \mathcal{G}$ of μ such that $|\int \varphi d\mu - \int \varphi d\nu| \leq \epsilon$ for any $\nu \in \mathcal{G}^\dagger$. Since X satisfies the almost specification property, Theorem 2.5 guarantees the existence of an ergodic measure $\nu \in \mathcal{G}^\dagger$ such that $h(\nu) \geq h(\mu) - \epsilon$.

Hence, by Propositions 2.4 and 3.2,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \Omega_n^{\varphi}(\mathcal{G}) &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(X) \cap \mathcal{E}_n^{-1}(\mathcal{G})} e^{S_n\varphi(x)} - P(\varphi) \\ &\geq h(\nu) + \int \varphi d\nu - P(\varphi) \\ &\geq h(\mu) + \int \varphi d\mu - P(\varphi) - 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, inequality (3.3) follows, completing the proof of Theorem A.

3.2. **Proof of Theorem B.** In this subsection we prove Theorem B. Its proof relies essentially on the following proposition, which provides a bridge between the density of periodic measures and the construction of suitable subshifts.

Proposition 3.3. Let $T\colon [0,1] \to [0,1]$ be as in Theorem B and let Σ_T denote the coding space of T. Let $\varphi \in C(\Sigma_T)$. Then for any $\epsilon > 0$, any $\mu \in \mathcal{M}_\sigma^e(\Sigma_T)$, and any neighborhood $\mathcal{G}$ of μ, there exists a subshift $Y \subset \Sigma_T$ such that

- Y satisfies the periodic specification property;
- $P(\varphi|_Y) \geq P(\varphi) - 2\epsilon$, where $\varphi|_Y$ denotes the restriction of φ with respect to Y;
- there exists an ergodic measure $\nu \in \mathcal{G}$ such that $h(\nu) \geq h(\mu) - \epsilon$ and $\nu(Y) = 1$.

Proof. Since σ is expansive, there is an ergodic equilibrium state m such that $P(\varphi) = h(m) + \int \varphi dm$. Choose $\delta > 0$ so small that $D(m, \rho) \leq 2\delta$ implies $|\int \varphi dm - \int \varphi d\rho| \leq \epsilon$. Choose $m^\diamond$ as follows: set $m = m^\diamond$ if $h(m) > 0$, and otherwise take $m^\diamond \in \mathcal{M}_\sigma^p(X)$ with $D(m, m^\diamond) \leq \delta$ if $h(m) = 0$ by using the assumption $\mathcal{M}_\sigma^e(\Sigma_T)$ is dense in $\mathcal{M}_\sigma^e(\Sigma_T)$. Then it is clear that $h(m) = h(m^\diamond)$, and by Lemma 2.6 $m^\diamond$ is liftable. Hence there is a $\bar{m^\diamond} \in \mathcal{M}_\sigma^e(\Sigma_\mathcal{D})$ such that $m^\diamond = \bar{m^\diamond} \circ \Psi^{-1}$. Since $\Sigma_\mathcal{D}$ is a topologically transitive countable Markov shift, it follows from Theorem 2.7 that there exist a finite set $\mathcal{F}_1 \subset \mathcal{D}$ and an ergodic measure $\bar{m^*}$ on $\Sigma_{\mathcal{F}_1}$ such that denoting $m^* := \bar{m^*} \circ \Psi^{-1}$, we have $D(m^\diamond, m^*) \leq \delta$ (hence $D(m, m^*) \leq 2\delta$), $h(m^*) \geq h(m^\diamond) - \epsilon = h(m) - \epsilon$ and $m^*(\Psi(\Sigma_{\mathcal{F}_1})) = 1$.

Similarly, we may find a finite subset $\mathcal{F}_2 \subset \mathcal{D}$ and an ergodic $\nu \in \mathcal{G}$ such that $h(\nu) \geq h(\mu) - \epsilon$ and $\nu(\Psi(\Sigma_{\mathcal{F}_2})) = 1$. Since the Markov diagram $\mathcal{D}$ possesses a unit loop, there is a vertex $C \in \mathcal{D}$ such that $C \to C$. Since $\mathcal{D}$ is irreducible and both $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite, there exists a finite irreducible subset $\mathcal{F} \subset \mathcal{D}$ which contains $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \{C\}$. Since $\mathcal{F}$ is finite and $C \in \mathcal{F}$, $Y := \Psi(\Sigma_{\mathcal{F}})$ is a topologically mixing sofic shift, which implies that Y satisfies the periodic specification property. Since $m^*(Y) \geq m^*(\Psi(\Sigma_{\mathcal{F}_1})) = 1$, we have $m^*(Y) = 1$. Hence the classical Variational Principle 2.2 implies that

$$P(\varphi|_Y) \geq h(m^*) + \int \varphi dm^* \geq h(m) + \int \varphi dm - 2\epsilon = P(\varphi) - 2\epsilon.$$

Similarly, we have $\nu(Y) = 1$, which proves the proposition. □

Remark 3.1. The main idea behind Proposition 3.3 can be summarized as follows. Given an ergodic invariant measure or an equilibrium state, we approximate it by periodic measures when necessary so that it becomes liftable to a countable Markov extension. This use of the density of periodic measures is essential: measures with positive entropy are liftable, whereas those with zero entropy are not in general, and periodic approximation guarantees liftability through Lemma 2.6. After this step, Takahasi's entropy-approximation theorem is applied to the lifted measures to construct a subshift Y with suitable properties. In particular, the density of periodic measures plays a crucial role in ensuring the success of this scheme.

We note that Propositions 3.4 and 3.5 are analogous to Propositions 2.2 and 3.2, which were used in the proof of Theorem A under the almost specification approach. Here, they serve as the corresponding ingredients for Theorem B based on the Markov diagram approach.

Proposition 3.4. Let T be as in Theorem B and let Σ_T denote the coding space of T. Then for any $\varphi \in C(\Sigma_T)$, we have

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_n(\Sigma_T)} e^{S_n\varphi(x)}.$$

Proof. The inequality $\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_n(\Sigma_T)} e^{S_n\varphi(x)} \leq P(\varphi)$ easily follows from the definition (2.1) of $P(\varphi)$.

On the other hand, by Proposition 3.3 for any $\epsilon > 0$, we can find a subshift $Y \subset \Sigma_T$ such that Y satisfies the periodic specification property and $P(\varphi|_Y) \geq P(\varphi) - 2\epsilon$. Since Y satisfies the periodic specification property, by Proposition 2.4, we have

$$P(\varphi|_Y) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_n(Y)} e^{S_n\varphi(x)}.$$

Hence

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n(\Sigma_T)}e^{S_n\varphi(x)}\geq\lim_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n(Y)}e^{S_n\varphi(x)}=P(\varphi|_Y)\geq P(\varphi)-2\epsilon,$$

which proves the proposition. □

Proposition 3.5. Let T be as in Theorem B and let Σ_T denote the coding space of T. Then for any $\mu\in\mathcal{M}_\sigma^e(\Sigma_T)$ and any open neighborhood $\mathcal{G}\subset\mathcal{M}(X)$ of μ, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n(\Sigma_T)\cap\mathcal{E}_n^{-1}(\mathcal{G})}e^{S_n\varphi(x)}\geq h(\mu)+\int\varphi d\mu.$$

Proof. By Proposition 3.3, there exist a subshift $Y\subset\Sigma_T$ and an ergodic measure $\nu\in\mathcal{G}$ such that Y satisfies the periodic specification property, $P(\varphi|_Y)\geq P(\varphi)-2\epsilon$, $h(\nu)\geq h(\mu)-\epsilon$ and $\nu(Y)=1$. Since Y satisfies the periodic specification property, Proposition 3.2 implies that

$$\liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n(Y)\cap\mathcal{E}_n^{-1}(\mathcal{G})}e^{S_n\varphi(x)}\geq h(\nu)+\int\varphi d\nu.$$

Hence we have

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n(\Sigma_T)\cap\mathcal{E}_n^{-1}(\mathcal{G})}e^{S_n\varphi(x)} &\geq \liminf_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n(Y)\cap\mathcal{E}_n^{-1}(\mathcal{G})}e^{S_n\varphi(x)}\\ &\geq h(\nu)+\int\varphi d\nu\\ &\geq h(\mu)+\int\varphi d\mu-2\epsilon,\end{aligned}$$

which proves the proposition. □

Now we give a proof of Theorem B. Let $T\colon[0,1]\to[0,1]$ be as in Theorem B and let Σ_T denote the coding space of T. The upper bound

$$\limsup_{n\to\infty}\frac{1}{n}\log\Omega_n^\varphi(\mathcal{K})\leq\sup_{\mu\in\mathcal{K}}q^\varphi(\mu)\qquad(\mathcal{K}\subset\mathcal{M}(\Sigma_T):\ \text{closed})$$

follows directly from Theorems 2.4 and 3.1.

Let $\mathcal{G}\subset\mathcal{M}(\Sigma_T)$ be an open subset and fix $\mu\in\mathcal{G}$. In what follows, we show that

$$\liminf_{n\to\infty}\frac{1}{n}\log\Omega_n^\varphi(\mathcal{G})\geq q^\varphi(\mu).\tag{3.4}$$

If μ is not σ-invariant, then $q^\varphi(\mu)=-\infty$ so we may, without loss of generality, assume that $\mu\in\mathcal{M}_\sigma(\Sigma_T)$. Given any $\epsilon>0$, choose a neighborhood $\mathcal{G}^\dagger\subset\mathcal{G}$ of μ such that $|\int\varphi d\mu-\int\varphi d\nu|\leq\epsilon$ for any $\nu\in\mathcal{G}^\dagger$. By the assumption of Theorem B, Proposition 2.8 implies that there exists an ergodic measure

$\nu \in \mathcal{G}^\dagger$ such that $h(\nu) \geq h(\mu) - \epsilon$. Thus it follows from Propositions 3.4 and 3.5 that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n}\log \Omega_n^\varphi(\mathcal{G}) &= \liminf_{n\to\infty} \frac{1}{n}\log \sum_{x\in P_n(\Sigma_T)\cap \mathcal{E}_n^{-1}(\mathcal{G})} e^{S_n\varphi(x)} - P(\varphi)\\ &\geq h(\nu) + \int \varphi d\nu - P(\varphi)\\ &\geq h(\mu) + \int \varphi d\mu - P(\varphi) - 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the desired inequality (3.4) follows, completing the proof of Theorem B.

4. Applications

In this section, we apply Theorems A and B to several concrete systems. We begin with the β-shift, a prototypical example beyond specification that admits applications of both approaches. This example illustrates the intersection between the two frameworks, while the following subsections focus separately on applications of Theorem A and Theorem B, respectively.

Example 4.1. (β-shifts). For $\beta > 1$, the *β-transformation* $T_\beta : [0,1] \to [0,1]$ is defined by

$$T_\beta(x) = \begin{cases} \beta x \ (\mathrm{mod} 1), & (x \neq 1),\\ \lim_{y\to 1-0}(\beta y \ (\mathrm{mod} 1)), & (x = 1),\end{cases}$$

and was introduced by Rényi [27]. The associated symbolic coding space Σ_β is called the *β-shift*.

It is well known that Σ_β fails to satisfy the specification property for Lebesgue-almost every $\beta > 1$ (see, e.g., [28]). Nevertheless, Thompson [36] showed that every β-shift satisfies the *periodic almost specification property*. Hence Theorem A applies to all β-shifts and yields a level-2 large deviation principle via periodic points.

Let $k \geq 1$ be a largest integer, which is less than β. To verify the assumptions of Theorem B, consider the Markov diagram $\mathcal{D}$ associated with the β-transformation. Although it is a well-known fact that $\mathcal{D}$ is irreducible, we are not aware of a specific reference that explicitly states this. For completeness, we briefly outline a proof below. Indeed, one can show that for any open interval $I \subset [0,1]$, there exist a positive integer n and an open subinterval $L \subset I$ such that $T_\beta^n|_L$ is strictly monotonic and continuous, and $T_\beta^n(L) = (0,1)$. Hence by [6, Lemma 4.1], $\mathcal{D}$ is irreducible. By the definition of T_β, the leftmost monotonicity interval forms a full-branch. Consequently, by the definition of the coding space, $\sigma[0] = \Sigma_\beta$, which implies $[0] \to [0]$; and hence the diagram possesses a unit loop. Here $[0]$ denotes the 1-cylinder of the corresponding symbolic space (rather than a bibliographic reference); the same notation will also be used in Examples 4.5 and 4.6. Moreover, by a classical result of Sigmund [30, Theorem (ii)], the set of periodic measures $\mathcal{M}_\sigma^p(\Sigma_\beta)$ is dense in the set of ergodic measures $\mathcal{M}_\sigma^e(\Sigma_\beta)$. Therefore, the assumptions of Theorem B are also fulfilled, and Theorem B applies to β-shifts as well.

Remark 4.1. Although the almost specification property for the systems considered here is proved in Thompson [33], a careful reading of the proofs therein shows that the periodic almost specification property also holds. We therefore omit a detailed proof of this fact. The same observation applies to the subsequent examples as well.

4.1. **Applications of Theorem A.** In this subsection, we further illustrate Theorem A through symbolic dynamical systems satisfying the periodic almost specification property. Beyond the β-shift discussed in Example 4.1, we consider representative examples such as the S-gap shifts and other related subshifts with non-unique equilibrium states. Such systems provide important models for analyzing non-uniform specification behavior and the non-uniqueness of equilibrium states in symbolic dynamics (see, for example, [1, 2, 7, 18, 21, 20]).

Example 4.2. (S-gap shifts). Let $S \subset \mathbb{N}_0$ be an infinite set. The *S-gap shift* is the two-sided subshift $\Sigma_S \subset \{0,1\}^{\mathbb{Z}}$ defined by the rule that the number of 0's between consecutive 1's is an integer in S. Equivalently, the language of Σ_S consists of all subblocks of words of the form

$$\text{“}0^{n_1}10^{n_2}1\cdots 0^{n_k}1\text{”} \quad (n_i \in S\ (1 \le i \le k),\ k \ge 1).$$

The system Σ_S is topologically mixing if and only if

$$\gcd\{\, n+1 : n \in S \,\} = 1,$$

and it satisfies the specification property if and only if

$$\sup_{i\ge 1}(m_{i+1} - m_i) < \infty, \quad \text{where } S = \{m_1 < m_2 < \cdots\}$$

(see [18, Example 3.4] for instance). Thus there exist many choices of S for which Σ_S is topologically mixing yet fails to have the specification property.

In fact, it is known that every topologically mixing S-gap shift possesses the periodic almost specification property (see [8, Remark 5.1] for instance). Hence, by Theorem A, every topologically mixing S-gap shift satisfies a level-2 large deviation principle via periodic points.

Remark 4.2. If the S-gap shift Σ_S is not topologically mixing, that is, $p = \gcd\{\, n+1 : n \in S \,\} > 1$, then every periodic orbit of Σ_S has minimal period divisible by p. In this case, the periodic-orbit LDP does not hold, since only multiples of p occur as periods.

Recent work of Takahasi [35] shows that the Dyck shift, a classical example of a subshift failing intrinsic ergodicity, nevertheless satisfies a level-2 large deviation principle via periodic points. His result highlights that strong large-deviation behavior can persist even when the system admits multiple measures of maximal entropy. In this context, Theorem A provides a complementary perspective. Indeed, Example 4.3 below yields another non-intrinsically ergodic subshift beyond specification, obtained from the construction of Kwietniak–Oprocha–Rams, and shows that the same periodic-orbit LDP holds in this broader setting.

Example 4.3. (A non–intrinsically ergodic subshift). We recall a construction due to Kwietniak–Oprocha–Rams [21] that yields a class of subshifts without the specification property and with multiple measures of maximal entropy, while still satisfying the periodic almost specification property.

Fix integers $p, q \geq 2$ and set

$$A := \{1, \dots, p\} \times \{0, \dots, q-1\} \ \cup \ \{(0,0)\},$$

which will be the alphabet. For a finite block

$$((a_1, b_1), \dots, (a_n, b_n)) \in A^n$$

we write it in the two-row form

$$\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}.$$

Let $\{R_n\}_{n\geq 1}$ be an increasing sequence with $R_n \subset \{1, \dots, n\}$. For each $n \geq 1$ define the family F_n of blocks of the form

$$\begin{bmatrix} 0 & a_1 & \cdots & a_n \\ 0 & b_1 & \cdots & b_n \end{bmatrix},$$

where $1 \leq a_i \leq p$, $0 \leq b_1, \dots, b_n \leq q-1$, and $b_j = 0$ whenever $j \in R_n$. Put

$$\mathcal{F} := \bigcup_{n\geq 1} F_n \ \cup \ \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

Next set

$$\mathcal{M} = \left\{ \begin{bmatrix} a & \dots & a \\ b_1 & \dots & b_n \end{bmatrix} : 1 \leq a \leq p, \ 0 \leq b_1, \dots, b_n \leq q-1, \ n \geq 1 \right\}$$

and define the language $\mathcal{L}_R$ to be the set of all subblocks of words of the form

$$\text{“}UV_1 \cdots V_k\text{”} \quad (U \in \mathcal{M}, \ V_1, \cdots, V_k \in \mathcal{F}, \ k \geq 1).$$

Finally, let

$$X_R := \{x \in A^{\mathbb{Z}} : \text{ every finite subblock of } x \text{ belongs to } \mathcal{L}_R\}.$$

Kwietniak–Oprocha–Rams show in [21, Proposition 4.6] that if, for example,

$$R_n = \{j \leq n : \ j = k^2 \text{ for some } k \in \mathbb{N}\},$$

then the subshift X_R satisfies the following properties:

(I) X_R does *not* have the specification property;
(II) X_R is not intrinsically ergodic (that is, there exist multiple ergodic measures of maximal entropy);

(III) X_R satisfies the periodic almost specification property.

Since property (III) provides the periodic almost specification hypothesis required in Theorem A, we may apply Theorem A to X_R.

This yields a nontrivial family of symbolic systems for which a level-2 large-deviation principle via periodic points holds despite the failure of both specification and intrinsic ergodicity; and thus provides a complementary example to recent results obtained for other non-intrinsically ergodic systems [35].

Example 4.4. (Full shift with multiple equilibrium states). In [20], Kucherenko, Quas and Wolf constructed continuous (non-Hölder) potentials on the two-sided full shift on two symbols which exhibit phase transitions and, in particular, admit more than one ergodic equilibrium state; see [20, Theorem 1].

Let φ be such a potential. Since the two-sided full shift has the periodic specification property, Theorem A applies to (X, φ). Thus, even though φ has multiple equilibrium states, a level-2 large deviation principle via periodic points nevertheless holds.

This shows that the results apply beyond the Hölder setting and remain valid even when the equilibrium state is not unique.

4.2. **Applications of Theorem B.** In this subsection, we illustrate Theorem B through symbolic dynamical systems beyond the classical β-shift. It is known that for Lebesgue almost every parameters $(\alpha, \beta) \in [0, 1) \times (1, \infty)$, the (α, β)-shift fails to satisfy the specification property, and that, except for continuous cases, the generalized β-shift also lacks specification for almost every $\beta > 1$ (see [4, Main Theorem] for details). Nevertheless, these systems satisfy the assumptions of Theorem B, and hence a level-2 large deviation principle holds via periodic points. These systems are natural extensions of the classical β-shift and thus provide representative examples where Theorem B can be effectively applied. They have also been studied intensively in recent years (see, for example, [5, 11, 12, 23, 31, 34]).

Example 4.5. ((α, β)-shifts). For $\beta > 1$ and $0 \leq \alpha < 1$, the (α, β)-*transformation* $T_{\alpha,\beta} : [0, 1] \to [0, 1]$ is defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \beta x + \alpha \pmod 1, & (x \neq 1), \\ \lim\limits_{y \to 1-0} (\beta y + \alpha \pmod 1), & (x = 1), \end{cases}$$

and was introduced by Parry [24]. Let $\Sigma_{\alpha,\beta}$ denote the symbolic coding space of $T_{\alpha,\beta}$. Clearly, $\Sigma_{0,\beta}$ coincides with the β-shift.

It is known that if $0 \leq \alpha < 1$ and $\beta > 2$, then the Markov diagram associated with $T_{\alpha,\beta}$ is irreducible (see, for instance, [6, Example 4.1]). In this case, the second monotonicity interval of $T_{\alpha,\beta}$, counting from the left, forms a full branch. Consequently, by the definition of the coding space, we have $\sigma[1] = \Sigma_{\alpha,\beta}$, which implies that $[1] \to [1]$; hence the diagram possesses a unit loop. Furthermore, it is known that $\mathcal{M}_\sigma^p(\Sigma_{\alpha,\beta})$ is dense in $\mathcal{M}_\sigma^e(\Sigma_{\alpha,\beta})$. Hence both assumptions (I) and (II) of Theorem B are satisfied. Therefore, by Theorem B, the (α,β)-shift $\Sigma_{\alpha,\beta}$ satisfies a level-2 large deviation principle via periodic points.

Example 4.6. (Generalized β-shifts). Let $\beta > 1$ and k be a smallest integer, which is greater than or equal to β, and fix $E = (E_1, \dots, E_k) \in \{+1, -1\}^k$. We partition $[0,1]$ into k intervals

$$I_1 := [0, 1/\beta),\ I_2 := [1/\beta, 2/\beta), \dots, I_k := [(k-1)/\beta, 1].$$

The generalized β-transformation $T_{\beta,E}\colon [0,1] \to [0,1]$ was introduced by Góra [14] and defined by

$$T_{\beta,E}(x) = \begin{cases} \beta x - i + 1 & (x \in I_i,\ E(i) = +1), \\ -\beta x + i & (x \in I_i,\ E(i) = -1). \end{cases}$$

It is straightforward to verify that $T_{\beta,E}$ is continuous if and only if $E_i \neq E_{i+1}$ for all $1 \le i < k$. As mentioned at the beginning of this subsection, when $T_{\beta,E}$ is discontinuous, that is, when $E_i = E_{i+1}$ for some i, the system fails to satisfy the specification property for Lebesgue almost every $\beta > 1$. If $E = (-1, \dots, -1)$, then we call $T_{\beta,E}$ a $(-\beta)$-transformation [17]. Let $\Sigma_{\beta,E}$ denote the coding space of $T_{\beta,E}$ and call it the generalized β-shift.

Let $\beta > 2$. Arguing similarly to the proof of [5, Proposition 3.14], we can show that for any open interval $I \subset [0,1]$, there exist a positive integer n and an open subinterval $L \subset I$ such that $T_{\beta,E}^n|_L$ is strictly monotonic and continuous, and $T^n(L) = (0,1)$. Therefore, by [6, Lemma 4.1], the Markov diagram associated with $T_{\beta,E}$ is irreducible. The leftmost monotonicity interval of $T_{\beta,E}$ forms a full branch. Consequently, by the definition of the coding space, we have $\sigma[0] = \Sigma_{\beta,E}$, which implies that $[0] \to [0]$; hence the diagram possesses a *unit loop*. Furthermore, it is shown in [29] that the set of periodic measures is dense in the set of ergodic measures of $\Sigma_{\beta,E}$. Hence both assumptions (I) and (II) of Theorem B are satisfied, and by Theorem B, the generalized β-shift $\Sigma_{\beta,E}$ satisfies a level-2 large deviation principle via periodic points.

Note on AI usage. The author used ChatGPT (GPT-5, OpenAI) solely for English proofreading and language refinement. The use of this AI tool did not influence the originality, mathematical content, or intellectual contribution of this research.

References

[1] Dastjerdi, Dawoud Ahmadi; Jangjoo, Somaye, Dynamics and topology of S-gap shifts, Topology Appl. 159 (2012), no. 10-11, 2654–2661.

[2] Baker, Simon; Ghenciu, Andrei E, Dynamical properties of S-gap shifts and other shift spaces, J. Math. Anal. Appl. 430 (2015), no. 2, 633–647.

[3] Bowen, Rufus, Some systems with unique equilibrium states, Math. Systems Theory 8 (1974/75), no. 3, 193–202.

[4] Buzzi, Jérôme, Specification on the interval, Trans. Amer. Math. Soc. 349 (1997), no. 7, 2737–2754.
[5] Carapezza, Leonard; López, Marco; Robertson, Donald, Unique equilibrium states for some intermediate beta transformations, Stoch. Dyn. 21 (2021), no. 6, Paper No. 2150035, 25 pp.
[6] Chung, Yong, Moo; Yamamoto, Kenichiro, Large deviation principle for piecewise monotonic maps with density of periodic measures, Ergodic Theory Dynam. Systems 43 (2023), no. 3, 861–872.
[7] Climenhaga, Vaughn; Thompson, Daniel J., Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors, Israel J. Math. 192 (2012), no. 2, 785–817.
[8] Climenhaga, Vaughn; Thompson, Daniel J.; Yamamoto, Kenichiro, Large deviations for systems with non-uniform structure, Trans. Amer. Math. Soc. 369 (2017), no. 6, 4167–4192.
[9] Comman, Henri; Rivera-Letelier, Juan, Large deviation principles for non-uniformly hyperbolic rational maps, Ergodic Theory Dynam. Systems 31 (2011), no. 2, 321–349.
[10] Denker, Manfred; Grillenberger, Christian; Sigmund, Karl, Ergodic theory on compact spaces, Lecture Notes in Math., Vol. 527 Springer-Verlag, Berlin-New York, 1976, iv+360 pp.
[11] Faller, Bastien, Contribution to the ergodic theory of piecewise monotone continuous map, Ph. D. thesis, École Polytechnique Fédérale de Lausanne, 2008.
[12] Faller, B.; Pfister, C.-E., A point is normal for almost all maps $\beta x + \alpha$ mod1 or generalized β-transformations, Ergodic Theory Dynam. Systems 29 (2009), no. 5, 1529–1547.
[13] Gelfert, Katrin; Wolf, Christian, On the distribution of periodic orbits, Discrete Contin. Dyn. Syst. 26 (2010), no. 3, 949–966.
[14] Góra, Paweł, Invariant densities for generalized β-maps, Ergodic Theory Dynam. Systems, 27 (5) (2007),1583–1598.
[15] Hofbauer, Franz, Intrinsic ergodicity of piecewise monotonic transformations with positive entropy II, Israel J. Math. 38 (1-2) (1981), 107–115.
[16] Hofbauer, Franz, Generic properties of invariant measures for simple piecewise monotonic transformations, Israel J. Math. 59 (1987), no. 1, 64–80.
[17] Ito, Shunji; Sadahiro, Taizo, Beta-expansions with negative bases, Integers 9 (2009), A22, 239–259.
[18] Jung, Uijin, On the existence of open and bi-continuing codes, Trans. Amer. Math. Soc. 363 (2011), no. 3, 1399–1417.
[19] Kifer, Yuri, Large deviations, averaging and periodic orbits of dynamical systems, Comm. Math. Phys. 162 (1994), no. 1, 33–46.
[20] Kucherenko, Tamara; Quas, Anthony; Wolf, Christian, Multiple phase transitions on compact symbolic systems, Adv. Math. 385 (2021), Paper No. 107768, 19 pp.
[21] Kwietniak, Dominik; Oprocha, Piotr; Rams, Michał, On entropy of dynamical systems with almost specification, Israel J. Math. 213 (2016), no. 1, 475–503.
[22] Lind, Douglas; Marcus, Brian, An introduction to symbolic dynamics and coding, Second edition Cambridge Math. Lib. Cambridge University Press, Cambridge, 2021. xix+550 pp.
[23] Oguchi, Mai; Shinoda, Mao, Hausdorff dimension of the parameters for (α, β)-shifts with the specification property, Dyn. Syst. 39 (2024), no. 4, 848–855.
[24] Parry, William, Representations for real numbers, Acta Math. Acad. Sci. Hungar. 15 (1964), 95–105.
[25] Pfister, Charles-Edouard; Sullivan, Wayne G, Large deviations estimates for dynamical systems without the specification property, Applications to the β-shifts. Nonlinearity 18 (2005), no. 1, 237–261.
[26] Pollicott, Mark; Sridharan, Shrihari, Large deviation results for periodic points of a rational map, J. Dyn. Syst. Geom. Theor. 5 (2007), no. 1, 69–77.
[27] Rényi, Alfréd, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungar. 8 (1957), 477–493.
[28] Schmeling, Jörg. Symbolic dynamics for β-shifts and self-normal numbers.
[29] Shinoda, Mao; Yamamoto, Kenichiro, Density of periodic measures and large deviation principle for generalised mod one transformations, Nonlinearity 37 (2024), no. 2, Paper No. 025003, 15 pp.
[30] Sigmund, Karl. On the distribution of periodic points for β-shifts. Monatsh. Math. 82 (1976), no. 3, 247–252.
[31] Suzuki, S., Artin–Mazur zeta functions of generalized β-transformations, Kyushu Journal of Mathematics, Vol. 71 (2017), No. 1, 85–103.

[32] Takahashi, Yōichirō, Entropy functional (free energy) for dynamical systems and their random perturbations, Stochastic analysis (Katata/Kyoto, 1982), 437–467, North-Holland Math. Library, 32, North-Holland, Amsterdam, 1984.

[33] Takahasi, Hiroki, Entropy approachability for transitive Markov shifts over infinite alphabet, Proc. Amer. Math. Soc. 148 (2020), no. 9, 3847–3857.

[34] Takahasi, Hiroki, Uniqueness of minimizer for countable Markov shifts and equidistribution of periodic points, J. Stat. Phys. 181 (2020), no. 6, 2415–2431.

[35] Takahasi, Hiroki, Large Deviation Principles for periodic points of the Dyck shift, arXiv:2503.14164, 2025.

[36] Thompson, D. J., Irregular sets, the β-shift and the almost specification property, Trans. Amer. Math. Soc. **364** (2012), 5395–5414.

[37] Thompson, D. J., Generalized β-transformations and the entropy of unimodal maps, Commentarii Mathematici Helvetici, Vol. 92 (2017), No. 4, 777–800.

[38] Yamamoto, Kenichiro, On the weaker forms of the specification property and their applications. Proc. Amer. Math. Soc. 137 (2009), no. 11, 3807–3814.

[39] Yamamoto, Kenichiro, Large deviations and periodic orbits of dynamical systems without the specification property, Shibaura Institute of Technology Research Bulletin, Science and Engineering Series, vol. 56, no. 1 (2012), 43-45 (non-refereed).

[40] Young, Lai-Sang, Some large deviation results for dynamical systems, Trans. Amer. Math. Soc. 318 (1990), no. 2, 525–543.

Department of General Education, Nagaoka University of Technology, Nagaoka 940-2188, JAPAN

Email address: k_yamamoto@vos.nagaokaut.ac.jp