Large Deviations and Periodic Orbits of Dynamical Systems without the Specification Property

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Abstract

For a positively expansive continuous map or an expansive homeomorphism of a compact metric space satisfying the strong almost product property, we prove a level-2 large deviation principle for the distribution of periodic points. This is a generalization of the result shown by Kifer.

Key words: large deviation, periodic points, specification property, almost product property

1 Introduction

This paper is devoted to the study of level-2 large deviation principles for dynamical systems. Let (X, d) be a compact metric space and f be a continuous map from X to itself. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X with the weak topology. We say that a sequence $\{\Omega_n\}$ of Borel probability measures on $\mathcal{M}(X)$ is said to satisfy a *level-2 large deviation principle with a rate function* $q\colon \mathcal{M}(X) \to [-\infty, 0]$ if q is upper semicontinuous,

$$\liminf_{n\to\infty} \frac{1}{n} \log \Omega_n(G) \geq \sup_{\mu\in G} q(\mu)$$

holds for any open set $G \subset \mathcal{M}(X)$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \Omega_n(F) \leq \sup_{\mu\in F} q(\mu)$$

holds for any closed set $F \subset \mathcal{M}(X)$. Usually, Ω_n is taken as the distribution of periodic points, iterated preimages, or Birkhoff averages. See [4] for the precise definitions of them.

Large deviation problems for dynamical systems have been extensively studied by several authors such as [3, 4, 5, 6](the case of periodic-points), [4](the case of iterated-preimages) and [2, 4, 5, 11](the case of Birkhoff averages), but mainly under the condition that the dynamical system has the specification property. The specification property was introduced by Bowen [1] and holds for typical chaotic dynamical systems including transitive Anosov diffeomorphisms and topologically mixing subshifts of finite type. However, there are many dynamics without the specification, such as β-shifts, ergodic automorphisms and generic non-hyperbolic dynamical systems. Recently, Pfister and Sullivan proved that all β-shifts satisfy the level-2 large deviation principle in the special case of Birkhoff averages.

The aim of this paper is to investigate the large deviation principle for β-shifts in the case of periodic points. For a continuous function φ on X, we define a sequence $\{\Omega_n^\varphi\}$ of Borel probability measures on $\mathcal{M}(X)$ as

$$\Omega_n^\varphi := \sum_{x\in P_n(f)} \frac{\exp(S_n\varphi(x))}{\sum_{z\in P_n(f)} \exp(S_n\varphi(z))} \delta_{\mathcal{E}_n(x)}.$$

Here $P_n(f) = \{x \in X : f^n(x) = x\}$, $S_n\varphi(x) = \sum_{j=0}^{n-1} \varphi(f^j(x))$, $\mathcal{E}_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$ and δ_y denotes the Dirac mass at the point $y \in X$. In this paper, we call $\{\Omega_n^\varphi\}$ the *distribution of periodic points* (for φ). Now we state our main theorem of this paper.

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Theorem 1.1. Let $f\colon X\to X$ be a positively expansive continuous map or an expansive homeomorphism of a compact metric space (X,d). If f satisfies the strong almost product property, then for any continuous function φ, Ω_n^φ satisfies a large deviation principle with a rate function $q^\varphi\colon \mathcal{M}(X)\to[-\infty,0]$ given by

$$q^\varphi(\mu)=\begin{cases} h_\mu(f)+\int\varphi d\mu - P(f,\varphi) & (\mu\in\mathcal{M}_f(X));\\ -\infty & (\text{otherwise}).\end{cases}$$

Here $h_\mu(f)$ denotes the *metric entropy* of $\mu\in\mathcal{M}_f(X)$ and $P(f,\varphi)$ denotes the *topological pressure* of φ.

We know that all β-shifts satisfy the assumption of Theorem 1.1 (see [7, Example]). Thus, all β-shifts satisfy the level-2 large deviation principle in the case of periodic distributions.

In §2, we recall some background materials from ergodic theory, and give a proof of Theorem 1.1 in §3.

2 Preliminaries

Let (X,d) be a compact metric space and $f\colon X\to X$ be a continuous map. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X with the weak topology and denote by $\mathcal{M}_f(X)$ the set of all f-invariant Borel probability measures on X. Let $C(X,\mathbb{R})$ be the Banach space of continuous real-valued functions of X with the sup norm $\|\cdot\|_\infty$. Since $C(X,\mathbb{R})$ is separable, there exists a countable set $\{\varphi_1,\varphi_2,\cdots\}$ which is dense in $C(X,\mathbb{R})$. For $\mu,\nu\in\mathcal{M}(X)$, we define

$$D(\mu,\nu):=\sum_{n=1}^{\infty}\frac{|\int\varphi_n d\mu-\int\varphi_n d\nu|}{2^{n+1}\|\varphi_n\|_\infty}.$$

Then D is a compatible metric for $\mathcal{M}(X)$ and $(\mathcal{M}(X),D)$ is compact. It is easy to see that $D(\mu,\nu)\le 1$ for any $\mu,\nu\in\mathcal{M}(X)$. For $\mu\in\mathcal{M}(X)$ and $\epsilon>0$, we set $\mathbb{B}(\mu,\epsilon):=\{\nu\in\mathcal{M}(X): D(\mu,\nu)\le\epsilon\}$.

A continuous map $f\colon X\to X$ is said to satisfy the *specification property* if for any $\epsilon>0$, there is an integer M_ϵ such that for any $k\ge1$ and k points $x_1,\cdots,x_k\in X$ and for any sequence of integers $0\le a_1\le b_1<a_2\le b_2<\cdots<a_k\le b_k$ with $a_i-b_{i-1}\ge M_\epsilon$ $(2\le i\le k)$, there is an $x\in X$ with $d(f^{a_i+j}(x),f^j(x_i))\le\epsilon$ $(0\le j\le b_i-a_i, 1\le i\le k)$. If such a point x can be chosen as a periodic point, then we say that f has the *strong specification property*. In [7], Pfister and Sullivan introduce a weaker form of the specification property, called almost product property.

Definition 2.1. We say that f has the *almost product property* if there exist $g\colon\mathbb{N}\to\mathbb{N}$ with $\lim_{n\to\infty}\frac{g(n)}{n}=0$ and $m\colon\mathbb{R}^+\to\mathbb{N}$ such that for any $k\ge1$, any $x_1,\cdots,x_k$, any $\epsilon_1>0,\cdots,\epsilon_k>0$ and any $n_1\ge m(\epsilon_1),\cdots,n_k\ge m(\epsilon_k)$, there exists $x\in X$ such that

$$\sharp\{0\le j\le n_i-1: d(f^{j+n_1+\cdots+n_{i-1}}(x),f^j(x_i))>\epsilon_i\}\le g(n_i)$$

holds for any $1\le i\le k$. Here $\sharp A$ denotes the cardinality of the set A. If such a point x can be chosen as a periodic point (i.e. $f^{n_1+\cdots+n_k}(x)=x$), then we say that f has the *strong almost product property.*

Example 2.1. Let $\beta>1$ and $F^\beta(x)=F(x)=\beta x$. $f\colon[0,1]\to[0,1]$ is then defined by

$$f(x)=\begin{cases}0 & (x=0);\\ 1 & (x\neq0, f(x)\in\mathbb{N});\\ F(x)\mod 1 & (\text{otherwise}).\end{cases}$$

We define a_i as $f^{-1}(\{0\})\backslash\{1\}=\{a_1,\cdots,a_p\}$ with $a_1<\cdots<a_p$ and set

$$I_0:=[0,a_1], I_1:=(a_1,a_2],\cdots,I_p:=(a_p,1].$$

Put $A:=\{0,\cdots,p\}$ and consider the shift space $(A^{\mathbb{Z}^+},\sigma)$. We define $(c_i)\in A^{\mathbb{Z}^+}$ as $c_i=j$ if and only if $f^i(1)\in I_j$ and set

$$X^\beta:=\{\omega\in A^{\mathbb{Z}^+}:\sigma^k(\omega)\le(c_i),\ k\ge0\}.$$

Then X^β is a shift invariant closed subset of the full shift. We call the restriction of σ into X^β the β-shift. It is known that the specification property holds only for a set of β of Lebesgue measure 0 ([9]), but for any $\beta>1$, β-shift satisfies the strong almost product property (see [7, Example]).

We review some known results which play an important role to prove Theorem 1.1.

Theorem 2.2. ([6, Theorem 2.1]) Let $f\colon X \to X$ be a continuous map of a compact metric space (X, d). If f satisfies the almost product property, then for any $\mu \in \mathcal{M}_f(X)$, any $h < h_\mu(f)$ and any neighborhood G of μ, there exists an ergodic measure $\nu \in G$ such that $h_\nu(f) > h$.

We say that a continuous map $f\colon X \to X$ is *positively expansive* if there exists $c > 0$ such that $d(f^n(x), f^n(y)) \le c$ $(n \ge 0)$ implies $x = y$. Similarly, a homeomorphism $f\colon X \to X$ is said to be *expansive* if there exists $c > 0$ such that $d(f^n(x), f^n(y)) \le c$ $(n \in \mathbb{Z})$ implies $x = y$.

Theorem 2.3. ([10, Theorem 1.2]) Let $f\colon X \to X$ be a positively expansive continuous map or an expansive homeomorphism of a compact metric space (X, d). If f satisfies the strong almost product property, then for any $\varphi \in C(X, \mathbb{R})$,

$$P(f, \varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(f)} e^{S_n\varphi(x)}$$

holds.

Let $\epsilon > 0$, $\delta > 0$ and $n \ge 1$. A subset E is called (δ, n, ϵ)*-separated* if for any two distinct points $x, y \in E$,

$$\sharp\{0 \le j \le n-1 : d(f^j(x), f^j(y)) > \epsilon\} > \delta n$$

holds.

Proposition 2.4. ([6, Proposition 2.1]) Let μ be ergodic and $h < h_\mu(f)$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that for any neighborhood F of μ in $\mathcal{M}(X)$, there exists $N \in \mathbb{N}$ so that for any $n \ge N$, there exists a (δ, n, ϵ)-separated set $\Gamma \subset X_{n,F}$ such that

$$\sharp\Gamma \ge e^{nh},$$

where $X_{n,F} := \{x \in X : \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \in F\}$.

Theorem 2.5. ([3, Theorem 6]) Let $f\colon X \to X$ be a positively expansive map or an expansive homeomorphism of a compact metric space (X, d). Then for any $\varphi \in C(X, \mathbb{R})$ and any closed set $F \subset \mathcal{M}_f(X)$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(f) \cap X_{n,F}} e^{S_n\varphi(x)} \le \sup_{\mu \in F} \left(h_\mu(f) + \int \varphi d\mu \right).$$

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

Proposition 3.1. Let $f\colon X \to X$ be a continuous map of a compact metric space (X, d) and $\varphi \in C(X, \mathbb{R})$. Suppose that f satisfies the strong almost product property. Then for any ergodic measure μ and any open neighborhood G of μ, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in P_n(f) \cap X_{n,G}} e^{S_n\varphi(x)} \ge h_\mu(f) + \int \varphi d\mu.$$

Proof. Given any $\eta > 0$, choose a sufficiently small $\gamma > 0$ such that $\mathbb{B}(\mu, 3\gamma) \subset G$ and for any $\nu \in \mathbb{B}(\mu, 3\gamma)$

$$\left| \int \varphi d\mu - \int \varphi d\nu \right| \le \eta$$

holds. It follows from Proposition 2.4 that there exist $\epsilon > 0$, $\delta > 0$ and $N \in \mathbb{N}$ such that for any $n \ge N$, there exists a (δ, n, ϵ)-separated set $\Gamma \subset X_{n,\mathbb{B}(\mu,3\gamma)}$ with $\sharp\Gamma \ge e^{n(h_\mu(f)-\eta)}$.

Since X is compact, we can find $\zeta > 0$ with $\zeta \le \frac{\epsilon}{2}$ such that $D(\delta_x, \delta_y) \le \gamma$ and $|\varphi(x) - \varphi(y)| \le \eta$ whenever $d(x, y) \le \zeta$. Let $m\colon \mathbb{R}^+ \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ be as in Definition 2.1. Choose an integer $n > 0$ so large that $n \geq \max\{N, m(\zeta)\}$ and $g(n) \leq \eta n$ hold. Then by the strong almost product property, for any $z \in \Gamma$, there exists a point $\sigma(z) \in P_n(f)$ such that

$$\sharp\Lambda_z \leq g(n),$$

where $\Lambda_z := \{0 \leq j \leq n-1 : d(f^j(z), f^j(\sigma(z))) > \zeta\}$. Then for any $z \in \Gamma$, we have

$$\begin{aligned} D(\mu, \mathcal{E}_n(\sigma(z))) &\leq D(\mu, \mathcal{E}_n(z)) + D(\mathcal{E}_n(z), \mathcal{E}_n(\sigma(z))) \\ &\leq \frac{1}{n}\sum_{j\in\Lambda_z} D(\delta_{f^j(z)}, \delta_{f^j(\sigma(z))}) + \frac{1}{n}\sum_{j\in\{0,\cdots,n-1\}\setminus\Lambda_z} D(\delta_{f^j(z)}, \delta_{f^j(\sigma(z))}) + \gamma \\ &\leq \frac{g(n)}{n} + 2\gamma \\ &\leq 3\gamma. \end{aligned}$$

This implies $\sigma(Z) \in X_{n,G}$. Since Γ is (δ, n, ϵ)-separated, the map $\sigma \colon \Gamma \to P_n(f) \cap X_{n,G}$ is injective. Moreover by the choice of ζ,

$$\begin{aligned} S_n\varphi(\sigma(z)) &= \sum_{j\in\Lambda_z} \varphi(f^j(\sigma(z))) + \sum_{j\in\{0,\cdots,n-1\}\setminus\Lambda_z} \varphi(f^j(\sigma(z))) \\ &\geq \sum_{j\in\Lambda_z} \varphi(f^j(z)) - 2g(n)\|\varphi\|_\infty + \sum_{j\in\{0,\cdots,n-1\}\setminus\Lambda_z} \varphi(f^j(z)) - \eta n \\ &\geq S_n\varphi(z) - 2\eta n \end{aligned}$$

holds for any $z \in \Gamma$. Therefore we have

$$\begin{aligned} \sum_{x\in P_n(f)\cap X_{n,F}} e^{S_n\varphi(x)} &\geq \sum_{z\in\Gamma} e^{S_n\varphi(\sigma(z))} \\ &\geq \sum_{z\in\Gamma} e^{S_n\varphi(z)-2\eta n} \\ &\geq \sum_{z\in\Gamma} e^{n(\int \varphi d\mu - 3\eta)} \\ &= \sharp\Gamma e^{n(\int \varphi d\mu - 3\eta)} \\ &\geq e^{n(h_\mu(f) + \int \varphi d\mu - 4\eta)}, \end{aligned}$$

which proves the proposition. □

Now we give a proof of Theorem 1.1. The upper estimate

$$\limsup_{n\to\infty} \frac{1}{n} \log \Omega_n^\varphi(F) \leq \sup_{\mu\in F} q^\varphi(\mu) \qquad (F \subset \mathcal{M}(X): \text{ closed})$$

is a directly consequence of Theorems 2.3 and 2.5.

Let $G \subset \mathcal{M}(X)$ be an open subset and $\mu \in G$. In what follows we will show that

$$\liminf_{n\to\infty} \frac{1}{n} \log \Omega_n^\varphi(G) \geq q^\varphi(\mu). \tag{3.1}$$

If μ is not f-invariant, $q^\varphi(\mu) = -\infty$ and so without loss of generality, we may assume that $\mu \in \mathcal{M}_f(X)$. Given any $\eta > 0$, choose a neighborhood $G' \subset G$ such that $|\int \varphi d\mu - \int \varphi d\nu| \leq \eta$ for any $\nu \in G'$. Since f satisfies the almost product property, it follows from Theorem 2.2 that there exists an ergodic measure $\nu \in G'$ such that $h_\nu(f) > h_\mu(f) - \eta$. Thus it follows from Theorem 2.3 and Proposition 3.1 that

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log \Omega_n^\varphi(G) &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_n(f)\cap X_{n,G}} e^{S_n\varphi(x)} - P(f,\varphi) \\ &\geq h_\nu(f) + \int \varphi d\nu - P(f,\varphi) \\ &\geq h_\mu(f) + \int \varphi d\mu - P(f,\varphi) - 2\eta. \end{aligned}$$

Since $\eta > 0$ is arbitrary, we have (3.1), which proves Theorem 1.1.

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