ERGODIC OPTIMIZATION FOR CONTINUOUS FUNCTIONS ON THE DYCK-MOTZKIN SHIFTS

MAO SHINODA, HIROKI TAKAHASI, KENICHIRO YAMAMOTO

Abstract. Ergodic optimization aims to describe dynamically invariant probability measures that maximize the integral of a given function. The Dyck and the Motzkin shifts are well-known examples of transitive subshifts that are not intrinsically ergodic. We show that the space of continuous functions on any Dyck-Motzkin shift splits into two subsets: one is a dense G_δ set with empty interior for which any maximizing measure has zero entropy; the other is contained in the closure of the set of functions having uncountably many, fully supported measures that are Bernoulli. One key ingredient of a proof of this result is the path connectedness of the space of ergodic measures of the Dyck-Motzkin shift.

1. Introduction

Ergodic optimization aims to describe properties of dynamically invariant maximizing measures. In its most basic form, main constituent components are: a continuous map T of a compact metric space X; the space $M(X,T)$ of T-invariant Borel probability measures endowed with the weak* topology together with the space $M^{\mathrm{e}}(X,T)$ of its ergodic elements; a continuous function $f\colon X\to\mathbb{R}$. Elements of $M(X,T)$ that attain the supremum

$$\Lambda_T(f)=\sup\left\{\int f\mathrm{d}\mu\colon \mu\in M(X,T)\right\} \tag{1.1}$$

are called *f-maximizing measures*. The set of f-maximizing measures, denoted by $M_{\max}(f)$, is non-empty and contains elements of $M^{\mathrm{e}}(X,T)$. For a given (X,T) and a Banach space of real-valued functions on X, we aim to establish properties of elements of $M_{\max}(f)$ for a 'typical' function f in the space. The regularity of functions is crucial. For (X,T) with some expanding or hyperbolic behavior and a Hölder continuous f, the Mañé-Conze-Guivarc'h lemma characterizes f-maximizing measures via their supports [1, 2, 9, 28]. The analysis of functions in the space $C(X)$ of real-valued continuous functions on X endowed with the supremum norm $\|\cdot\|_{C^0}$ is completely different: the Mañé-Conze-Guivarc'h lemma is no longer valid, but duality arguments are available and can be used to prove the occurrence of pathological phenomena.

Morris [29, Corollary 2] proved that for (X,T) with Bowen's specification property [4], the maximizing measure is unique, fully supported on X (charging any non-empty open subset of X), has zero entropy, and is not strongly mixing for a

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generic continuous function, thereby unifying the the result of Bousch and Jenkinson [3] and that of Brémont [6]. In contrast to Morris's result, Shinoda [31, Theorem A] proved that for a dense set of continuous functions on a topologically mixing Markov shift (subshift of finite type), there exist uncountably many, fully supported ergodic maximizing measures with positive entropy, which are actually Bernoulli. For an analogous result on expanding Markov interval maps with holes, see [32].

As a generalization and unification of the result of Morris [29, Corollary 2] and that of Shinoda [31, Theorem A] concerning entropies and supports of maximizing measures, in [33] the authors proved the following statement for a wide class of non-Markov subshifts Σ over a finite alphabet: There exists a constant $h_{\mathrm{spec}}^{\perp}(\Sigma) \in [0, h_{\mathrm{top}}(\Sigma))$ such that if $h_{\mathrm{spec}}^{\perp}(\Sigma) \leq H < h_{\mathrm{top}}(\Sigma)$ then

(I) the set $\{f \in C(\Sigma)\colon h(\mu) \leq H \text{ for all } \mu \in M_{\max}(f)\}$ is dense G_δ;

(II) for any $f \in C(\Sigma)$ not contained in the dense G_δ set in (I) and for any neighborhood U of f in $C(\Sigma)$, there exists $g \in U$ such that $\{\mu \in M_{\max}(g)\colon h(\mu) > H\}$ contains uncountably many fully supported ergodic measures,

where $h_{\mathrm{top}}(\Sigma) < \infty$ denotes the topological entropy of Σ, and $h(\mu) \in [0, h_{\mathrm{top}}(\Sigma)]$ denotes the measure-theoretic entropy of a shift-invariant measure μ on Σ. The constant $h_{\mathrm{spec}}^{\perp}(\Sigma)$ is called *the obstruction entropy to specification* [8]. Recall that dense G_δ sets are countable intersections of open dense subsets, and a property that holds for a dense G_δ set is said to be generic.

A subshift carrying a unique measure of maximal entropy is called *intrinsically ergodic*. The statement in the previous paragraph requires $h_{\mathrm{spec}}^{\perp}(\Sigma) < h_{\mathrm{top}}(\Sigma)$ that actually implies the intrinsic ergodicity of the subshift Σ [7, Theorem C]. Therefore, it is natural to ask if an analogous statement holds for a subshift that is not intrinsically ergodic.

As a counterexample to the conjecture of Weiss [37], Krieger [21] proved that the Dyck shift has exactly two ergodic measures of maximal entropy. The Motzkin shift [25, 26] is a subshift determined by the Dyck shift and the units. Krieger [22] introduced a certain class of shift spaces having some algebraic property, called property A subshifts. The Dyck and Motzkin shifts are prototypes in this class, and rich sources of interesting phenomena different from those in Markov shifts, see [16, 25, 26, 27, 36] for example. In this paper we consider ergodic optimization for continuous functions on these subshifts that are not intrinsically ergodic.

1.1. **Ergodic optimization for continuous functions.** In Section 2.2 we will introduce shift spaces Σ_D with two non-negative integer parameters (M, N), called the *Dyck-Motzkin shifts*. The $(M, 0)$ Dyck-Motzkin shift is nothing but the Dyck shift on $2M$ symbols, consisting of M brackets, left and right in pair, whose admissible words are words of legally aligned brackets. The (M, N) Dyck-Motzkin shift with $N \neq 0$ is nothing but the Motzkin shift on $2M + N$ symbols, consisting of the M brackets and N units whose admissible words are words of legally aligned brackets with freely interspersed units. Our main result below recovers the above (I) (II) with $H = 0$ for the Dyck-Motzkin shifts.

Theorem A. *Let Σ_D be a Dyck-Motzkin shift.*

(a) *The set* $\{f \in C(\Sigma_D)\colon h(\mu) = 0$ *for all* $\mu \in M_{\max}(f)\}$ *is dense* G_δ.
(b) *There exists a dense subset* $\mathscr{D}$ *of* $C(\Sigma_D)$ *such that for any* $f \in \mathscr{D}$, $M_{\max}(f)$ *contains uncountably many elements that are fully supported on* Σ_D *and Bernoulli.*

Statements like Theorem A replacing 'Bernoulli' in (b) by 'ergodic, positive entropy' were obtained in [33, Theorem B] for any subshift Σ for which $h_{\rm top}(\Sigma) > h^{\perp}_{\rm spec}(\Sigma) = 0$ and ergodic measures are entropy dense. The definition of entropy density can be found in [12]. For the (M, N) Dyck-Motzkin shift Σ_D, we note that $h_{\rm top}(\Sigma_D) = h^{\perp}_{\rm spec}(\Sigma_D) = \log(M + N + 1)$ holds, and ergodic measures are not entropy dense.

For proofs of (a) and (b) of Theorem A, we develop ideas of Morris [29, Theorem 1.1, Corollary 2] and Shinoda [31, Theorem A] respectively, both related to approximations of ergodic measures in the weak* topology. Regarding (a), a key observation is that Bowen's specification property does not hold for the Dyck-Motzkin shift, but the hypothesis of Bowen's specification property in [29, Corollary 2] can actually be weakened to the density of invariant measures of zero entropy in the space of ergodic measures. For any Dyck-Motzkin shift, we show in Section 2.7 that CO-measures (shift-invariant ergodic measures supported on periodic orbits) are dense in the space of ergodic measures. This property allows us to slightly modify the argument in the proof of [29, Theorem 1.1] to conclude Theorem A(a).

The proof of Theorem A(b) deserves a special attention as it gives a new insight into the structure of the spaces of ergodic measures of the Dyck-Motzkin shifts. Below we give further explanations, but first require simple definitions. Let X be a topological space and let x, $y \in X$ be distinct points. A continuous map $p\colon [0, 1] \to X$ such that $p(0) = x$ and $p(1) = y$ is called a *path* joining x, y. We say a path $p\colon [0, 1] \to X$ *lies* in $Y \subset X$ if $p(t) \in Y$ holds for all $t \in [0, 1]$.

Israel [19, Section V] proved an approximation theorem about tangent functionals to convex functions, and used it for lattice systems in statistical mechanics to prove the existence of a dense set of continuous interactions that admit uncountably many ergodic equilibrium states. Shinoda's proof of [31, Theorem A] is an adaptation of Israel's argument to ergodic optimization that is briefly outlined as follows. For a real-valued continuous function f_0 on a compact metric space X and a continuous map $T\colon X \to X$ such that $M^{\rm e}(X, T)$ is arcwise connected, she took an ergodic measure $\mu \in M_{\max}(f_0)$ and a path $t \in [0, 1] \mapsto \mu_t \in M^{\rm e}(X, T)$ such that $\mu_0 = \mu$, and then used the version of the Bishop-Phelps theorem [19, Theorem V.1.1] to approximate f_0 by $f \in C(X)$ so that $\{\mu_t\colon t \in [0, 1]\}$ contains uncountably many elements of $M_{\max}(f)$. One can control properties of the maximizing measures by carefully choosing the path. To choose a path such that the uncountably many maximizing measures are fully supported and Bernoulli, Shinoda used Sigmund's result [35, Theorem B] which asserts that the space of shift-invariant ergodic mesures on a topologically mixing Markov shift is path connected.

Our strategy for the proof of Theorem A(b) is to substantially extend Shinoda's path argument to the Dyck-Motzkin shifts. Since the Dyck-Motzkin shifts are not

Markov, Sigmund's result [35, Theorem B] is no longer valid. To overcome this difficulty, we delve into the structure of the shift space and show the following abundance of paths of ergodic measures of high complexity:

(1) *Any pair of ergodic measures of the Dyck-Motzkin shift in any weak* open ball can be joined by a path that lies in that ball*, and moreover
(2) *this path 'almost' lies in the set of measures that are fully supported and Bernoulli.*

Since any Dyck-Motzkin shift contains many subshifts of finite type (SFTs) in its shift space, Sigmund's result [35, Theorem B] can still be used to find a high complexity path joining two ergodic measures for the Dyck-Motzkin shift that are supported on the same properly embedded SFT. Proper embedding means a one-to-one, into but not onto conjugacy of shift spaces. It is not always possible to find an SFT that supports a given pair of ergodic measures. Hamachi and Inoue [16, Theorem 5.3] provided a necessary and sufficient condition for the existence of a proper embedding of an irreducible SFT into the Dyck shift in terms of topological entropy of the shift spaces and multipliers of periodic points in them. From their result it immediately follows that for any integer $M \geq 2$, there is no SFT of entropy $\log(M+1)$ that can be embedded into the Dyck shift on $2M$ symbols. For a corresponding result on the property A subshifts, see [17, Theorem 5.8]. In particular, there is no embedded SFT in the Dyck shift that supports the two ergodic measures of maximal entropy $\log(M+1)$ constructed by Krieger [21]. An analogous statement holds for the Motzkin shift.

Krieger's construction of the two ergodic measures of maximal entropy for the Dyck shift relies on the construction of two different Borel embeddings of the full shift on $M+1$ symbols into the Dyck shift. Borel embedding means a one-to-one conjugacy defined on a shift-invariant proper Borel subset that does not have a continuous extension to the whole shift space (see Section 2.6 for the definition). In order to prove (1) (2), we begin by extending Krieger's construction [21, Section 4] of Borel embeddings of the full shifts to the Dyck-Motzkin shift Σ_D. We then transport sequences of ergodic measures on Σ_D to the two full shift spaces via the inverses of these Borel embeddings, and construct paths joining the transported measures. Finally, we transport these paths back to the space of ergodic measures on Σ_D, and concatenate the transported paths to obtain a desired path. Since the Borel embeddings do not have continuous extensions to the whole full shift spaces, the transport in the last step needs a justification. For precise statements of (1) (2) with relevant definitions and more detailed explanations, see Section 3.1.

1.2. **Connectedness of spaces of ergodic measures.** Recall that a topological space X is *path connected* if for any pair x, y of its distinct points there exists a path that lies in X and joins x, y. We say X is *arcwise connected* if it is path connected and the path can be taken to be a homemorphism onto its image. We say X is *locally path connected* (resp. *locally arcwise connected*) if any point in it has a neighborhood base consisting of open sets that are path connected (resp. arcwise connected). For a Hausdorff space, the path connectedness implies the arcwise connectedness.

The property (1) immediately yields the following result.

Theorem B. *The space of shift-invariant ergodic Borel probability measures on any Dyck-Motzkin shift is path connected and locally path connected with respect to the weak* topology.*

An investigation of the topological structure of the space $M(X,T)$ began with the works of Sigmund [34, 35] in the 70s, and has recently gained a renewed impetus. One motivation comes from the fact that every Polish topological space is homeomorphic to a set of ergodic measures of some shift space over a finite alphabet, which follows from [18, Theorem] and [11, Theorem 5]. The space $M(X,T)$ is a Choquet simplex whose extreme points are precisely the set $M^{\mathrm{e}}(X,T)$ of ergodic measures. Choquet simplices with dense extreme points are isomorphic up to affine homeomorphisms, and the unique simplex is called the Poulsen simplex [30]. If $M(X,T)$ is a Poulsen simplex, then the path connectedness and the local path connectedness of $M^{\mathrm{e}}(X,T)$ follow from a complete description of the topological structure of the Poulsen simplex given in [24].

The space of ergodic measures of any Dyck-Motzkin shift is not Poulsen, and it is path connected by Theorem B. In his blog, Climenhaga gave a rough sketch of a proof of the path connectedness of the space of ergodic measures of the Dyck shift. However a justification is needed. For other examples of subshifts whose spaces of ergodic measures are not Poulsen and path connected, see [20, Corollary 7] and [23, Section 4]. For relevant results on structures of the spaces of ergodic measures of partially hyperbolic diffeomorphisms, see [10, 15]. A sufficient condition for the path connectedness of $M^{\mathrm{e}}(X,T)$ in terms of periodic points and CO-measures of T was given in [14, Theorem 6.1], which however does not apply to the Dyck-Motzkin shift. In any of these previous works, there was no discussion on the local path connectedness of the space of ergodic measures. Needless to say, a proof of the local path connectedness is more delicate than that of the mere path connectedness.

The rest of this paper consists of three sections. In Section 2 we collect preliminary results needed for the proofs of our main results. In Section 3 we give precise statements of (1) (2) and prove them. In Section 4 we complete the proofs of Theorems A and B.

2. Preliminaries

In this section we collect preliminary results needed for the proofs of our main results. After recalling basic terms and notation in symbolic dynamics in Section 2.1, we introduce the Dyck-Motzkin shifts in Section 2.2. Following Krieger [21, Section 4], in Section 2.3 we classify ergodic measures of the Dyck-Motzkin shifts into three types, and in Section 2.4 construct embeddings of two full shifts on $M+N+1$ symbols into the (M,N) Dyck-Motzkin shift. In Section 2.5 we deal with a transportation of ergodic measures by means of these embeddings. In Section 2.6 we show that the embeddings constructed in Section 2.4 are Borel embeddings. In Section 2.7 we prove an approximation result by CO-measures for the Dyck-Motzkin shifts.

2.1. **Basic terms and notation.** Let S be a non-empty finite discrete set, called a *finite alphabet* and let $S^{\mathbb{Z}}$ denote the two-sided Cartesian product topological space of S. The left shift acts continuously on $S^{\mathbb{Z}}$. A shift-invariant closed subset of $S^{\mathbb{Z}}$ is called a *subshift* over S. A finite string $\omega = \omega_1\omega_2\cdots\omega_n$ of elements of S is called a *word* of length n in S. For convenience, we introduce an empty word $\emptyset$ by the rules $\emptyset\omega = \omega\emptyset = \omega$ for any word ω in S. The word length of the empty word is set to be 0. For a subshift Σ over S and $n \in \mathbb{N}\cup\{0\}$, let $\mathcal{L}_n(\Sigma)$ denote the collection of words in S of word length n that appear in some elements of Σ. Put $\mathcal{L}(\Sigma) = \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{L}_n(\Sigma)$. Words in $\mathcal{L}(\Sigma)\setminus\{\emptyset\}$ are called *admissible*. For a subshift Σ and for $j \in \mathbb{Z}$, $n \in \mathbb{N}$, $\omega \in \mathcal{L}_n(\Sigma)$, define

$$\Sigma(j;\omega) = \{(x_i)_{i\in\mathbb{Z}} \in \Sigma\colon x_{i+j-1} = \omega_i \ \text{ for } i = 1,\ldots,n\}.$$

Unless otherwise stated, we use the letter σ to denote the left shift acting on a subshift Σ: $(\sigma x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. For a subshift Σ, let $M(\Sigma)$ denote the space of shift-invariant Borel probability measures on Σ endowed with the weak* topology, and let $M^{\mathrm{e}}(\Sigma)$ denote the space of elements of $M(\Sigma)$ that are ergodic. For a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of Borel probability measures on Σ that converges to $\mu \in M(\Sigma)$ in the weak* topology, we write $\mu_n \to \mu$.

Let $t \in \mathbb{N}$ and let $A = (a_{ij})$ be a $t\times t$ matrix all whose entries are 0 or 1. We assume that every row of A has a nonzero entry. The subshift

$$\Sigma_A = \{(x_i)_{i\in\mathbb{Z}} \in \{1,\ldots,t\}^{\mathbb{Z}}\colon a_{x_ix_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}$$

is called a *Markov shift* or a *subshift of finite type* (SFT) determined by the transition matrix A. In the case $a_{ij} = 1$ for all i, $j \in \{1,\ldots,t\}$, Σ_A is called the *full shift* on t symbols. Write $A^n = (a_{ij}^{(n)})$ for $n \in \mathbb{N}$. We say Σ_A is *topologically mixing* if there exists $n \in \mathbb{N}$ such that $a_{ij}^{(n)} \neq 0$ for all i, $j \in \{1,\ldots,t\}$.

2.2. **The Dyck-Motzkin shift.** Let $M \geq 2$, $N \geq 0$ be integers and let

$$D_\alpha = \{\alpha_1,\ldots,\alpha_M\} \ \text{ and } \ D_\beta = \{\beta_1,\ldots,\beta_M\},$$

which are interpreted as sets of left brackets and right brackets respectively: α_i and β_i are in pair for $i = 1,\ldots,M$. The (M,N) Dyck-Motzkin shift is a two-sided subshift over the finite alphabet

$$D = D_\alpha \cup D_0 \cup D_\beta,$$

which consists of $2M+N$ symbols where $\#D_0 = N$. If $N = 0$ then D_0 is an empty set. If $N \neq 0$ then we write $D_0 = \{1_1,\ldots,1_N\}$. Let D^* denote the set of finite words in D. Consider the monoid with zero, with $2M+N$ generators in D and the unit element 1 with relations

$$\alpha_i\cdot\beta_j = \delta_{i,j},\ 0\cdot 0 = 0,\ 1_k\cdot 1_\ell = 1 \ \text{ for } i,j \in \{1,\ldots,M\},\ k,\ell \in \{1,\ldots,N\},$$
$$\omega\cdot 1 = 1\cdot\omega = \omega,\ \omega\cdot 0 = 0\cdot\omega = 0 \ \text{ for } \omega \in D^*\cup\{1\},$$
$$\omega\cdot 1_\ell = 1_\ell\cdot\omega = \omega \ \text{ for } \omega \in D^* \text{ and } \ell \in \{1,\ldots,N\},$$

where $\delta_{i,j}$ denotes Kronecker's delta. For $n \in \mathbb{N}$ and $\omega_1\cdots\omega_n \in D^*$ let

$$\mathrm{red}(\omega_1\cdots\omega_n) = \prod_{i=1}^n \omega_i.$$

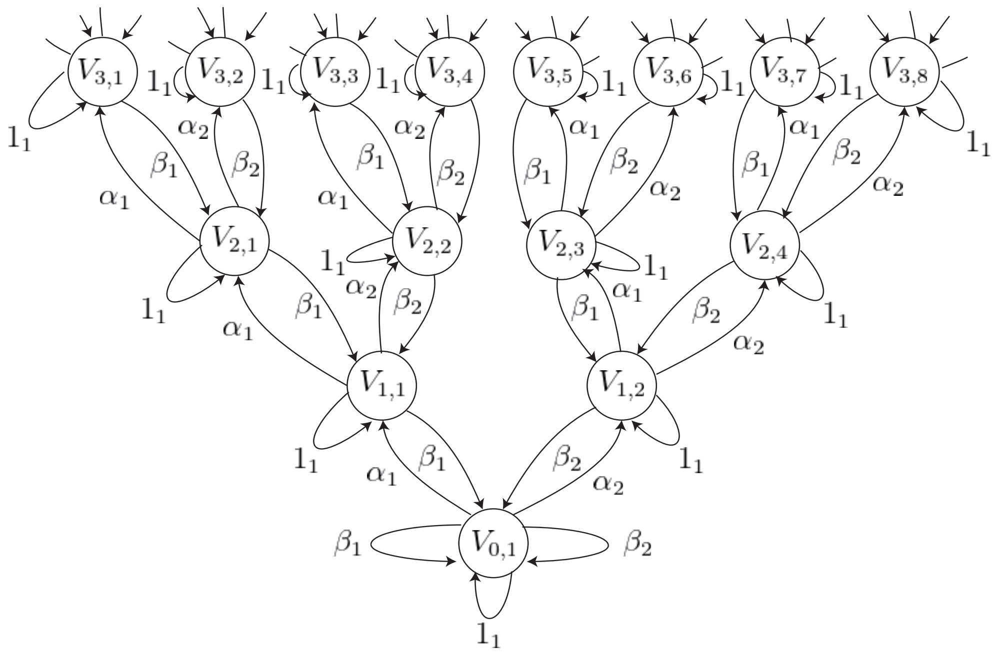

FIGURE 1. Part of the labeled directed graph associated with the $(2, 1)$ Dyck-Motzkin shift. Each upward (resp. downward) edge is labeled with α_1 or α_2 (resp. β_1 or β_2). Each vertex has one loop edge labeled with 1_1.

The (M, N) Dyck-Motzkin shift is defined by

$$\Sigma_D = \{x = (x_i)_{i\in\mathbb{Z}} \in D^{\mathbb{Z}} \colon \mathrm{red}(x_j \cdots x_k) \neq 0 \ \text{ for all } j, k \in \mathbb{Z} \text{ with } j < k\}.$$

Another way to define the (M, N) Dyck Motzkin shift Σ_D is the following. Consider a labeled directed graph that consists of infinitely many vertices $V_{i,j}$, $i = 0, 1, \ldots,$ $j = 1, \ldots, M^i$ together with edges each labeled with a unique symbol in D. Each vertex $V_{i,j}$ has M outgoing edges

$$V_{i,j} \xrightarrow{\alpha_1} V_{i+1,Mj-M+1}, \ \ldots, \ V_{i,j} \xrightarrow{\alpha_{M-1}} V_{i+1,Mj-1}, \ V_{i,j} \xrightarrow{\alpha_M} V_{i+1,Mj}$$

and M incoming edges

$$V_{i,j} \xleftarrow{\beta_1} V_{i+1,Mj-M+1}, \ \ldots, \ V_{i,j} \xleftarrow{\beta_{M-1}} V_{i+1,Mj-1}, \ V_{i,j} \xleftarrow{\beta_M} V_{i+1,Mj}.$$

The bottom vertex $V_{0,1}$ has additional M loop edges $V_{0,1} \xrightarrow{\beta_1,\ldots,\beta_M} V_{0,1}$. If $N \neq 0$, then each vertex $V_{i,j}$ has additional N loop edges $V_{i,j} \xrightarrow{1_1,\ldots,1_N} V_{i,j}$. Let Σ_D^+ denote the set of one-sided infinite sequences of elements of D that are associated with the infinite labeled paths in this graph starting at $V_{0,1}$. Then Σ_D is the invertible extension of Σ_D^+:

$$\Sigma_D = \{x = (x_i)_{i\in\mathbb{Z}} \in D^{\mathbb{Z}} \colon x_j x_{j+1} \cdots \in \Sigma_D^+ \text{ for all } j \in \mathbb{Z}\}.$$

Part of the labeled directed graph associated with the $(2, 1)$ Dyck-Motzkin shift is shown in FIGURE 1. The set of admissible words coincides with the set of strings of labels associated with the finite paths in the graph starting at $V_{0,1}$. Removing all the loop edges labeled with 1_1 gives part of the labeled directed graph associated with the $(2,0)$ Dyck-Motzkin shift. It is easy to see that Σ_D has infinitely many forbidden words, and so it is not a Markov shift.

2.3. **Classification of ergodic measures.** For each $j \in \mathbb{Z}$ define $G_j \colon \Sigma_D \to \{-1,0,1\}$ by

$$G_j(x) = \sum_{k=1}^{M} (\delta_{\alpha_k, x_j} - \delta_{\beta_k, x_j}).$$

We have $G_j(x) = 1$ if $x_j \in D_\alpha$, $G_j(x) = 0$ if $x_j \in D_0$, $G_j(x) = -1$ if $x_j \in D_\beta$. For each $i \in \mathbb{Z}$ define $H_i \colon \Sigma_D \to \mathbb{Z}$ by

$$H_i(x) = \begin{cases} \sum_{j=0}^{i-1} G_j(x) & \text{for } i \geq 1, \\ -\sum_{j=i}^{-1} G_j(x) & \text{for } i \leq -1, \\ 0 & \text{for } i = 0. \end{cases}$$

The function H_i for $i \geq 1$ (resp. $i \leq -1$) counts the difference between the number of symbols in D_α and that in D_β appearing in $x_0 \cdots x_{i-1}$ (resp. $x_i \cdots x_{-1}$). For i, $j \in \mathbb{Z}$ define

$$\{H_i = H_j\} = \{x \in \Sigma_D \colon H_i(x) = H_j(x)\}.$$

For $i, j \in \mathbb{Z}$ with $i < j$, we have $H_i(x) = H_j(x)$ if and only if the number of symbols in D_α that appear in $x_i \cdots x_{j-1}$ equals the number of symbols in D_β that appear in $x_i \cdots x_{j-1}$. In particular, if $\mathrm{red}(x_i \cdots x_{j-1}) = 1$ then $H_i(x) = H_j(x)$ holds. If $x \in \Sigma_D$, $i < j$ and $H_i(x) = H_j(x)$ then $i + 1 < j$ holds. If moreover $x_i \in D_\alpha$ (resp. $x_{j-1} \in D_\beta$), then there exists $k \in \{i+1, \ldots, j-1\}$ (resp. $k \in \{i, \ldots, j-2\}$) such that the left bracket at position i (resp. the right bracket at position $j-1$) in x is closed with the corresponding right (resp. left) bracket at position k in x: $\mathrm{red}(x_i \cdots x_k) = 1$ and $\mathrm{red}(x_i \cdots x_\ell) \neq 1$ for all $\ell \in \{i, \ldots, k-1\}$ (resp. $\mathrm{red}(x_k \cdots x_{j-1}) = 1$ and $\mathrm{red}(x_\ell \cdots x_{j-1}) \neq 1$ for all $\ell \in \{k+1, \ldots, j-1\}$).

We introduce three pairwise disjoint shift-invariant Borel sets

$$A_0 = \bigcap_{i=-\infty}^{\infty} \left(\left(\bigcup_{j=1}^{\infty} \{H_{i+j} = H_i\} \right) \cap \left(\bigcup_{j=1}^{\infty} \{H_{i-j} = H_i\} \right) \right),$$

$$A_\alpha = \left\{ x \in \Sigma_D \colon \lim_{i \to \infty} H_i(x) = \infty \ \text{ and } \ \lim_{i \to -\infty} H_i(x) = -\infty \right\},$$

$$A_\beta = \left\{ x \in \Sigma_D \colon \lim_{i \to \infty} H_i(x) = -\infty \ \text{ and } \ \lim_{i \to -\infty} H_i(x) = \infty \right\}.$$

Note that all the three sets are dense in Σ_D. The next lemma classifies elements of the set $M^{\mathrm{e}}(\Sigma_D)$ of shift-invariant ergodic measures on Σ_D.

Lemma 2.1. *If $\mu \in M^{\mathrm{e}}(\Sigma_D)$, then either $\mu(A_0) = 1$, $\mu(A_\alpha) = 1$ or $\mu(A_\beta) = 1$.*

Proof. For the Dyck shift, the statement had been proved in [21, pp.102–103]. We treat the Dyck-Motzkin shift with a simpler argument. With the notation in

Section 2.1, for $x \in \Sigma_D$ let

$$l(x) = \sup\left\{ i \in \mathbb{Z} \colon x \in \bigcup_{k=1}^{M} \left(\Sigma_D(i;\alpha_k) \cap \bigcup_{j=1}^{\infty} \{H_{i+j} = H_i\}^c \right) \right\},$$

and

$$r(x) = \inf\left\{ i \in \mathbb{Z} \colon x \in \bigcup_{k=1}^{M} \left(\Sigma_D(i;\beta_k) \cap \bigcup_{j=1}^{\infty} \{H_{i-j+1} = H_i\}^c \right) \right\},$$

where the upper indices c denote the complement in Σ_D. For $i, j \in \mathbb{Z}$ let

$$A_{i,j} = \{x \in \Sigma_D \colon l(x) = i,\ r(x) = j\}.$$

If $x \in \Sigma_D \setminus (A_0 \cup A_\alpha \cup A_\beta)$, then both $l(x)$ and $r(x)$ are finite. Hence we have

$$\Sigma_D \setminus (A_0 \cup A_\alpha \cup A_\beta) \subset \bigcup_{i,j \in \mathbb{Z}} A_{i,j}.$$

For all $i, j \in \mathbb{Z}$ we have $\sigma A_{i,j} = A_{i-1,j-1}$, and so $A_{i,j} = A_{0,j-i} \neq \emptyset$. If $\mu \in M(\Sigma_D)$ then we have $\mu(A_{i,j}) = 0$. Since A_0, A_α, A_β are shift-invariant, if μ is ergodic then either $\mu(A_0) = 1$, $\mu(A_\alpha) = 1$ or $\mu(A_\beta) = 1$. □

2.4. **Construction of embeddings of the full shift.** We introduce two full shift spaces on $M + N + 1$ symbols over different sub-alphabets of D:

$$\Sigma_\alpha = (D_\alpha \cup D_0 \cup \{\beta\})^{\mathbb{Z}} \quad \text{and} \quad \Sigma_\beta = (\{\alpha\} \cup D_0 \cup D_\beta)^{\mathbb{Z}}.$$

Let σ_α, σ_β denote the left shifts acting on Σ_α, Σ_β respectively. With the notation in Section 2.1 we introduce two shift-invariant Borel sets of Σ_D:

$$B_\alpha = \bigcap_{i=-\infty}^{\infty} \bigcup_{k=1}^{M} \bigcup_{\ell=1}^{N} \left(\Sigma_D(i;\alpha_k) \cup \Sigma_D(i;1_\ell) \cup \left(\Sigma_D(i;\beta_k) \cap \bigcup_{j=1}^{\infty} \{H_{i-j+1} = H_{i+1}\} \right) \right),$$

$$B_\beta = \bigcap_{i=-\infty}^{\infty} \bigcup_{k=1}^{M} \bigcup_{\ell=1}^{N} \left(\Sigma_D(i;\beta_k) \cup \Sigma_D(i;1_k) \cup \left(\Sigma_D(i;\alpha_k) \cap \bigcup_{j=1}^{\infty} \{H_{i+j} = H_i\} \right) \right).$$

The set B_α (resp. B_β) is precisely the set of sequences in Σ_D such that any right (resp. left) bracket in the sequence is closed. One can check that

$$A_0 \cup A_\alpha \subset B_\alpha \quad \text{and} \quad A_0 \cup A_\beta \subset B_\beta.$$

Define $\phi_\alpha \colon \Sigma_D \to \Sigma_\alpha$ by

$$(\phi_\alpha(x))_i = \begin{cases} \beta & \text{if } x_i \in D_\beta, \\ x_i & \text{otherwise.} \end{cases}$$

In other words, $\phi_\alpha(x)$ is obtained by replacing all β_k, $k \in \{1, \ldots, M\}$ in x by β. Clearly ϕ_α is continuous, not one-to-one. Similarly, define $\phi_\beta \colon \Sigma_D \to \Sigma_\beta$ by

$$(\phi_\beta(x))_i = \begin{cases} \alpha & \text{if } x_i \in D_\alpha, \\ x_i & \text{otherwise.} \end{cases}$$

In other words, $\phi_\beta(x)$ is obtained by replacing all α_k, $k \in \{1, \ldots, M\}$ in x by α. Clearly ϕ_β is continuous, not one-to-one. We set

$$K_\alpha = \phi_\alpha(B_\alpha) \text{ and } K_\beta = \phi_\beta(B_\beta).$$

For each $j \in \mathbb{Z}$ define $G_{\alpha,j} \colon \Sigma_\alpha \to \{-1, 0, 1\}$ by

$$G_{\alpha,j}(x) = \sum_{k=1}^{M} (\delta_{\alpha_k, x_j} - \delta_{\beta, x_j}).$$

For each $i \in \mathbb{Z}$ define $H_{\alpha,i} \colon \Sigma_D \to \mathbb{Z}$ by

$$H_{\alpha,i}(x) = \begin{cases} \sum_{j=0}^{i-1} G_{\alpha,j}(x) & \text{for } i \geq 1, \\ -\sum_{j=i}^{-1} G_{\alpha,j}(x) & \text{for } i \leq -1, \\ 0 & \text{for } i = 0. \end{cases}$$

The function $H_{\alpha,i}$ for $i \geq 1$ (resp. $i \leq -1$) counts the difference between the number of symbols in D_α and that of β appearing in $x_0 \cdots x_{i-1}$ (resp. $x_i \cdots x_{-1}$). We define $\psi_\alpha \colon K_\alpha \to D^{\mathbb{Z}}$ by

$$(\psi_\alpha(y))_i = \begin{cases} \beta_k & \text{if } y_i = \beta,\ y_{s_\alpha(i,y)} = \alpha_k,\ k \in \{1, \ldots, M\}, \\ y_i & \text{otherwise,} \end{cases}$$

where

$$s_\alpha(i, y) = \max\{j < i + 1 \colon H_{\alpha,j}(y) = H_{\alpha,i+1}(y)\}.$$

Clearly ψ_α is continuous.

Similarly, for each $j \in \mathbb{Z}$ define $G_{\beta,j} \colon \Sigma_\beta \to \{-1, 0, 1\}$ by

$$G_{\beta,j}(x) = \sum_{k=1}^{M} (\delta_{\alpha, x_j} - \delta_{\beta_k, x_j}).$$

For each $i \in \mathbb{Z}$ define $H_{\beta,i} \colon \Sigma_D \to \mathbb{Z}$ by

$$H_{\beta,i}(x) = \begin{cases} \sum_{j=0}^{i-1} G_{\beta,j}(x) & \text{for } i \geq 1, \\ -\sum_{j=i}^{-1} G_{\beta,j}(x) & \text{for } i \leq -1, \\ 0 & \text{for } i = 0. \end{cases}$$

We define $\psi_\beta \colon K_\beta \to D^{\mathbb{Z}}$ by

$$(\psi_\beta(y))_i = \begin{cases} \alpha_k & \text{if } y_i = \alpha,\ y_{s_\beta(i,y)} = \beta_k,\ k \in \{1, \ldots, M\}, \\ y_i & \text{otherwise,} \end{cases}$$

where

$$s_\beta(i, y) = \min\{j > i \colon H_{\beta,j}(y) = H_{\beta,i}(y)\}.$$

Clearly ψ_β is continuous too.

Lemma 2.2. *For each $\gamma \in \{\alpha, \beta\}$ the following statements hold:*

(a) *$\psi_\gamma(K_\gamma) = B_\gamma$, and ψ_γ is a homeomorphism whose inverse is $\phi_\gamma|_{B_\gamma}$.*
(b) *$\phi_\gamma \circ \sigma|_{B_\gamma} = \sigma_\gamma \circ \phi_\gamma|_{B_\gamma}$ and $\sigma^{-1} \circ \psi_\gamma = \psi_\gamma \circ \sigma_\gamma^{-1}|_{K_\gamma}$.*

Proof. For each $\gamma \in \{\alpha, \beta\}$, it is straightforward to check that $\psi_\gamma \circ \phi_\gamma(x) = x$ for all $x \in B_\gamma$, and $\phi_\gamma \circ \psi_\gamma(y) = y$ for all $y \in K_\gamma$, which verifies (a). A proof of (b) is also straightforward. $\square$

2.5. **Transport of ergodic measures.** By Lemma 2.2, if $\gamma \in \{\alpha, \beta\}$ and $\nu \in M(\Sigma_\gamma)$ satisfies $\nu(K_\gamma) = 1$, then ν can be transported to a shift-invariant measure on Σ_D. The lemma below gives a sufficient condition for measures in $M^{\mathrm{e}}(\Sigma_\gamma)$ to have measure 1 on K_γ. For each $\gamma \in \{\alpha, \beta\}$ we set

$$\Sigma_\gamma(\gamma) = \{\omega \in \Sigma_\gamma \colon \omega_0 \in D_\gamma\} \quad \text{and} \quad \Sigma_\gamma(0) = \{\omega \in \Sigma_\gamma \colon \omega_0 \in D_0\},$$

and define $E_\gamma \colon \Sigma_\gamma \to \mathbb{R}$ by

$$E_\gamma = 2 \cdot \mathbb{1}_{\Sigma_\gamma(\gamma)} + \mathbb{1}_{\Sigma_\gamma(0)},$$

where $\mathbb{1}_{(\cdot)}$ denotes the indicator function. Note that E_γ is continuous. If $N = 0$ then $E_\gamma = 2 \cdot \mathbb{1}_{\Sigma_\gamma(\gamma)}$. If $\nu \in M(\Sigma_\alpha)$ and $\int E_\alpha \mathrm{d}\nu > 1$ (resp. $\nu \in M(\Sigma_\beta)$ and $\int E_\beta \mathrm{d}\nu > 1$), then ν gives more mass to the union of cylinders corresponding to the symbols in D_α (resp. D_β) than the cylinder corresponding to β (resp. α).

Lemma 2.3. *For each $\gamma \in \{\alpha, \beta\}$ the following statements hold:*
(a) *If $\nu \in M^{\mathrm{e}}(\Sigma_\gamma)$ and $\int E_\gamma \mathrm{d}\nu > 1$ then $\nu(K_\gamma) = 1$.*
(b) *K_γ is a dense subset of Σ_γ.*

Proof. The statements for the Dyck shift were essentially proved in [21, Section 4]. We extend the proofs there to the Dyck-Motzkin shift. By the definitions of B_α and ϕ_α we have

$$K_\alpha = \bigcap_{i=-\infty}^{\infty} \bigcup_{k=1}^{M} \bigcup_{\ell=1}^{N} \left(\Sigma_\alpha(i; \alpha_k) \cup \Sigma_\alpha(i; 1_\ell) \cup \left(\Sigma_\alpha(i; \beta) \cap \bigcup_{j=1}^{\infty} \{H_{\alpha, i-j+1} = H_{\alpha, i+1}\} \right) \right),$$

and by De Morgan's laws,

$$K_\alpha^c = \bigcup_{i=-\infty}^{\infty} \bigcap_{k=1}^{M} \bigcap_{\ell=1}^{N} \left(\Sigma_\alpha(i; \alpha_k)^c \cap \Sigma_\alpha(i; 1_\ell)^c \cap \left(\Sigma_\alpha(i; \beta)^c \cup \bigcap_{j=1}^{\infty} \{H_{\alpha, i-j+1} = H_{\alpha, i+1}\}^c \right) \right),$$

where the upper indices c denote the complements in Σ_α. For each $y \in K_\alpha^c$ there exists $i \in \mathbb{Z}$ such that $y_i = \beta$ and $H_{\alpha, i-j+1}(y) \neq H_{\alpha, i+1}(y)$ for all $j \in \mathbb{N}$. By induction, for all $j \geq 1$ we have

$$\#\{m \in \{i-j+1, \ldots, i\} \colon y_m = \beta\} > \#\{m \in \{i-j+1, \ldots, i\} \colon y_m \in D_\alpha\}.$$

If $\nu \in M^{\mathrm{e}}(\Sigma_\alpha)$ and $\int E_\alpha \mathrm{d}\nu > 1$, then $\nu(\Sigma_\alpha(0; \beta)) < \nu(\Sigma_\alpha(\alpha))$, and Birkhoff's ergodic theorem applied to $(\sigma_\alpha^{-1}, \nu)$ yields $\nu(K_\alpha) = 1$ as required in (a) for $\gamma = \alpha$. One can treat the case $\gamma = \beta$ analogously, using σ_β instead of σ_α^{-1}.

In order to prove (b), let ν_γ denote the Bernoulli measure on Σ_γ with the uniform distribution over $M + N + 1$ symbols, which are of entropy $\log(M + N + 1)$. Since $\int E_\gamma \mathrm{d}\nu_\gamma > 1$ by hypothesis, we have $\nu_\gamma(K_\gamma) = 1$ by (a). Since ν_γ is fully supported on Σ_γ, K_γ is a dense subset of Σ_γ. The proof of Lemma 2.3 is complete. $\square$

2.6. **Borel embeddings.** Let Σ_1, Σ_2 be two subshifts. For $i = 1, 2$, let σ_i denote the left shift acting on Σ_i. Let K be a shift-invariant, proper Borel subset of Σ_1. We say $\psi\colon K \to \Sigma_2$ is a *Borel embedding* of Σ_1 into Σ_2 if the following hold:

(i) ψ is continuous, injective and $\psi^{-1}\colon \psi(K) \to K$ is continuous,
(ii) $\sigma_2 \circ \psi = \psi \circ \sigma_1$,
(iii) there is no continuous map $\bar{\psi}\colon \Sigma_1 \to \Sigma_2$ such that $\bar{\psi} = \psi$ on K.

By [16, Theorem 5.3] and [17, Theorem 5.8], there is no proper embedding of the full shift on $M + N + 1$ symbols to Σ_D. From this and Lemma 2.2 it follows that K_γ is a proper subset of Σ_γ. Moreover the following holds.

Proposition 2.4. *For each $\gamma \in \{\alpha, \beta\}$ the map $\psi_\gamma\colon K_\gamma \to B_\gamma \subset \Sigma_D$ is a Borel embedding of Σ_γ into Σ_D.*

Proof. Conditions (i), (ii) in the definition of Borel embedding is a consequence of Lemma 2.2. It is left to show (iii). If there were a continuous map $\bar{\psi}_\gamma\colon \Sigma_\gamma \to \Sigma_D$ such that $\bar{\psi}_\gamma(y) = \psi_\gamma(y)$ for all $y \in K_\gamma$, then $\bar{\psi}_\gamma \circ \phi_\gamma\colon \Sigma_D \to \Sigma_D$ would be continuous. Since $\psi_\gamma \circ \phi_\gamma(x) = x$ holds for all $x \in B_\gamma$, $\bar{\psi}_\gamma \circ \phi_\gamma(x) = x$ would hold for all $x \in B_\gamma$. Since B_γ is dense in Σ_D Lemma 2.3(b), $\bar{\psi}_\gamma \circ \phi_\gamma(x) = x$ would hold for all $x \in \Sigma_D$. Then ϕ_γ would be injective, a contradiction. □

2.7. **Approximation of ergodic measures by CO-measures.** Let Σ be a subshift. A point $x \in \Sigma$ is called a *periodic point* of period $n \in \mathbb{N}$ if $\sigma^n x = x$. An element of $M(\Sigma)$ that is supported on the orbit of a single periodic point is called a *CO-measure*. Clearly CO-measures are ergodic. Let $M^{CO}(\Sigma)$ denote the set of CO-measures.

Recall that there exist three pairwise disjoint shift-invariant Borel subsets A_0, A_α, $A_\beta \subset \Sigma_D$ such that any shift-invariant ergodic measure has measure 1 to one of these three sets (Section 2.3 and Lemma 2.1). Let

$$M^{\mathrm{e}}_\gamma(\Sigma_D) = \{\mu \in M^{\mathrm{e}}(\Sigma_D)\colon \mu(A_\gamma) = 1\} \quad \text{for } \gamma \in \{0, \alpha, \beta\}.$$

Proposition 2.5. *For all $\gamma \in \{\alpha, \beta\}$ we have*

$$M^{\mathrm{e}}_\gamma(\Sigma_D) \subset \overline{M^{\mathrm{e}}_\gamma(\Sigma_D) \cap M^{CO}(\Sigma_D)}$$

and

$$M^{\mathrm{e}}_0(\Sigma_D) \subset \overline{M^{\mathrm{e}}_\gamma(\Sigma_D) \cap M^{CO}(\Sigma_D)}.$$

In particular, $M^{CO}(\Sigma_D)$ is dense in $M^{\mathrm{e}}(\Sigma_D)$.

Proof. For $\omega = \omega_1 \cdots \omega_k \in D^*$ and $n \in \mathbb{N}$, let $\omega^n \in D^*$ denote the n-fold concatenation: $(\omega^n)_i = \omega_j$, $j = i \bmod k$ for $i = 1, \ldots, kn$. Let $\omega \in D^*$. If $\mathrm{red}(\omega) = 1$ then we say ω is *neutral*. If $\mathrm{red}(\omega)$ is a concatenation of symbols in D_α (resp. D_β), then we say ω is *negative* (resp. *positive*). If ω is neutral, negative or positive then ω^n is admissible for all $n \in \mathbb{N}$.

In order to prove the first inclusion, let $\mu \in M^{\mathrm{e}}_\gamma(\Sigma_D)$. By Birkhoff's ergodic theorem, there exists $x \in A_\gamma$ such that $n^{-1}\sum_{i=0}^{n-1}\delta_{\sigma^i x} \to \mu$ and $n^{-1}\sum_{i=0}^{-n+1}\delta_{\sigma^i x} \to \mu$, where $\delta_{\sigma^i x}$ denotes the unit point mass at $\sigma^i x$. If $\gamma = \alpha$, then by the definition of A_α, for any $n \in \mathbb{N}$ there exists $l \in \mathbb{Z}$ such that $l \le -n$ and $\omega = x_l \cdots x_n$ is negative. Then $\Sigma_D(l; x_l \cdots x_n)$ contains exactly one periodic point of period $n - l + 1$. Let

μ_n denote the CO-measure supported on the orbit of this periodic point. Since $\{x\} = \bigcap_{n=1}^{\infty} \Sigma_D(l; x_l \cdots x_n)$ we have $\mu_n \in M_\alpha^{\mathrm{e}}(\Sigma_D)$ and $\mu_n \to \mu$. A proof for the case $\gamma = \beta$ is analogous.

In order to prove the second inclusion, let $\mu \in M_0^{\mathrm{e}}(\Sigma_D)$. By Birkhoff's ergodic theorem, there exists $x \in A_0$ such that $n^{-1} \sum_{i=0}^{n-1} \delta_{\sigma^i x} \to \mu$ and $n^{-1} \sum_{i=0}^{-n+1} \delta_{\sigma^i x} \to \mu$. By the definition of A_0, for any $n \in \mathbb{N}$ there exist l, $m \in \mathbb{Z}$ such that $l \leq -n$, $m \geq n$ and $\omega = x_l \cdots x_m$ is neutral. For all $k \in \mathbb{N}$, $\omega^{2k}\alpha_1$ is admissible, negative and $\Sigma_D(-k|\omega| + l; \omega^{2k}\alpha_1)$ contains exactly one periodic point of period $2k|\omega| + 1$. Let μ_n denote the CO-measure supported on the orbit of this periodic point. Clearly we have $\mu_n \in M_\alpha^{\mathrm{e}}(\Sigma_D)$. Since $\{x\} = \bigcap_{n=1}^{\infty} \Sigma_D(-k|\omega| + l; \omega^{2k})$, we obtain $\mu_n \to \mu$. This verifies the second inclusion for $\gamma = \alpha$. A proof for $\gamma = \beta$ is analogous, with all α_1 replaced by β_1. □

3. On the abundance of high complexity paths

The aim of this section is to prove the abundance of high complexity paths of ergodic measures for the Dyck-Motzkin shift informally stated in (1) (2) in Section 1. In Section 3.1 we give relevant definitions, and give a precise statement of this in Proposition 3.1. After proving two preliminary lemmas in Section 3.2, we complete the proof of Proposition 3.1 in Section 3.3. In Section 3.4 we comment more on the path connectedness of spaces of ergodic measures.

3.1. **A precise statement.** A path $t \in [0,1] \mapsto \mu_t \in M^{\mathrm{e}}(\Sigma_D)$ is called a *high complexity path* if the following two conditions hold:

- (A1) μ_t is fully supported on Σ_D and is Bernoulli for all $t \in [0,1]$ but countably many values;
- (A2) For any $\mu \in \{\mu_t \colon t \in [0,1]\}$, the set $\{t \in [0,1] \colon \mu_t = \mu\}$ is countable.

The next proposition is a key ingredient in the proof of Theorem A(b) and that of Theorem B.

Proposition 3.1 (The abundance of high complexity paths)**.** *Let U be a convex open subset of $M(\Sigma_D)$, let $\gamma \in \{\alpha, \beta\}$ and let μ^+, $\mu^- \in U \cap (M_0^{\mathrm{e}}(\Sigma_D) \cup M_\gamma^{\mathrm{e}}(\Sigma_D))$ be distinct measures. There exists a high complexity path that lies in $U \cap (M_0^{\mathrm{e}}(\Sigma_D) \cup M_\gamma^{\mathrm{e}}(\Sigma_D))$ and joins μ^+, μ^-.*

From Proposition 3.1 we immediately obtain the following statement.

Proposition 3.2. *The spaces $M_0^{\mathrm{e}}(\Sigma_D) \cup M_\alpha^{\mathrm{e}}(\Sigma_D)$ and $M_0^{\mathrm{e}}(\Sigma_D) \cup M_\beta^{\mathrm{e}}(\Sigma_D)$ are path connected and locally path connected.*

Proof. The path connectedness is a consequence of Proposition 3.1 with $U = M(\Sigma_D)$. Since any point in $M(\Sigma_D)$ has a neighborhood base consisting of convex open sets, the local path connectedness is also a consequence of Proposition 3.1. □

A proof of Proposition 3.1 has been inspired by the works of Sigmund [34, 35] on the path connectedness of the space of ergodic measures for Axiom A diffeomorphisms (essentially topologically mixing Markov shifts). In [34] he proved that the set of CO-measures are dense in the space of shift-invariant Borel probability measures. Later in the proof of [35, Theorem B] he showed that any pair of CO-measures can be joined by a path of ergodic measures, in such a way that if the two CO-measures lie in a convex open set, then the whole path can be chosen to lie in this set. These paths can be concatenated to form a path joining a given pair of ergodic measures.

The rest of this section except Section 3.4 is dedicated to a proof of Proposition 3.1 that breaks into three steps. We only give a proof for $\gamma = \alpha$ since that for $\gamma = \beta$ is identical. Using Proposition 2.5, we take sequences $\{\mu_n^+\}_{n\in\mathbb{N}}$, $\{\mu_n^-\}_{n\in\mathbb{N}}$ of CO-measures in U that approximate μ^+, μ^- respectively. Then we transport $\{\mu_n^+\}_{n\in\mathbb{N}}$, $\{\mu_n^-\}_{n\in\mathbb{N}}$ via $\phi_\alpha|_{B_\alpha} \colon B_\alpha \to K_\alpha \subset \Sigma_\alpha$, and apply Sigmund's result in the proof of [35, Theorem B] (see Lemma 3.5) to obtain sequences of paths that appropriately join the transported measures. Finally we transport these paths back to Σ_D via $\psi_\alpha \colon K_\alpha \to B_\alpha \subset \Sigma_D$, and concatenate all of them to form a path joining μ^+, μ^- with the required property. The transport in the last step needs to be justified since ψ_α is a Borel embedding of Σ_α into Σ_D as in Proposition 2.4, which does not have a continuous extension to the whole shift space Σ_α.

Remark 3.3. Gorodetski and Pesin [15] further developed Sigmund's argument explained as above to prove the path connectedness of some basic pieces of the space of ergodic measures for some partially hyperbolic diffeomorphisms. In [15], they did not treat the path connectedness of the whose space of ergodic measures.

3.2. **Preliminary lemmas.** Recall that $\sigma_\alpha K_\alpha = K_\alpha$ but K_α is not a subshift as it is not closed. With a slight abuse of notation, let $M(K_\alpha)$ denote the space of $\sigma_\alpha|_{K_\alpha}$-invariant Borel probability measures on K_α endowed with the weak* topology. Note that

$$M(K_\alpha) = \{\nu \in M(\Sigma_\alpha) \colon \nu(K_\alpha) = 1\}.$$

By Lemma 2.2(a), $\phi_\alpha|_{B_\alpha} \colon B_\alpha \to K_\alpha$ is a homeomorphism whose inverse is the Borel embedding $\psi_\alpha \colon K_\alpha \to B_\alpha$. Define a push-forward $\psi_\alpha^* \colon M(K_\alpha) \to M(\Sigma_D)$ by

$$\psi_\alpha^*(\nu) = \nu \circ \psi_\alpha^{-1}.$$

Since ψ_α is continuous, ψ_α^* is continuous.

Lemma 3.4. *If $\nu \in M(K_\alpha)$ is fully supported on Σ_α, then $\psi_\alpha^*(\nu)$ is fully supported on Σ_D.*

Proof. Since ψ_α is a homeomorphism by Lemma 2.2(a) and K_α is dense in Σ_α by Lemma 2.3(b), if $\nu \in M(K_\alpha)$ is fully supported on Σ_α then $\nu \circ \psi_\alpha^{-1}(B_\alpha) = \nu(K_\alpha) = 1$. Since B_α is dense in Σ_D, $\psi_\alpha^*(\nu)$ is fully supported on Σ_D. □

The next lemma is essentially due to Sigmund, shown in the proof of [35, Theorem B]. Here we only add a supplementary proof.

Lemma 3.5. *Let Σ be a topologically mixing Markov shift. Let μ, $\mu' \in M^{\mathrm{e}}(\Sigma)$ be distinct CO-measures, and let U be a convex open subset of $M^{\mathrm{e}}(\Sigma)$ that contains μ, μ'. There is a homeomorphism $t \in [0,1] \mapsto \theta_t \in U$ onto its image that is a high complexity path joining μ, μ'.*

Proof. It was shown in the proof of [35, Theorem B] that there exist a topologically mixing Markov shift Σ', a homeomorphism $\tau\colon \Sigma \to \Sigma'$ commuting with the shifts, and a path $t \in [0,1] \mapsto \theta_t \in U$ joining μ, μ' such that $\theta_t \circ \tau^{-1}$ is a Markov measure for all $t \in (0,1)$. In particular, $\theta_t \circ \tau^{-1}$ is a Gibbs state [5] for all $t \in (0,1)$, and it is Bernoulli by [5, Theorem 1.25] and [13]. Hence, θ_t is fully supported on Σ and is Bernoulli for all $t \in (0,1)$. A close inspection into the proof of [35, Theorem B] shows the injectivity of $t \in [0,1] \mapsto \theta_t \in U$. $\square$

3.3. **Proof of Proposition 3.1.** We fix a countable dense subset $\{f_n \not\equiv 0\colon n \in \mathbb{N}\}$ of $C(\Sigma_\alpha)$, and define a metric d on $M(\Sigma_\alpha)$ by

$$d(\nu,\nu') = \sum_{n=1}^{\infty} \frac{|\int f_n \mathrm{d}\nu - \int f_n \mathrm{d}\nu'|}{2^n \|f_n\|_{C^0}} \quad \text{for } \nu,\nu' \in M(\Sigma_\alpha).$$

The weak* topology on $M(\Sigma_\alpha)$ is metrizable by the metric d. For $\nu \in M(\Sigma_\alpha)$ and $\delta > 0$, let $U_\alpha(\nu,\delta)$ denote the open ball of radius δ about ν with respect to d.

Let U be a convex open subset of $M(\Sigma_D)$ and let μ^+, $\mu^- \in U \cap (M_0^{\mathrm{e}}(\Sigma_D) \cup M_\alpha^{\mathrm{e}}(\Sigma_D))$ be distinct measures. The definition of E_α gives

$$\int E_\alpha \circ \phi_\alpha \mathrm{d}\mu^+ \geq 1 \quad \text{and} \quad \int E_\alpha \circ \phi_\alpha \mathrm{d}\mu^- \geq 1. \tag{3.1}$$

Since U is open and ψ_α^* is continuous, there exists $q \in \mathbb{N}$ such that

$$\psi_\alpha^* \left(\left(U_\alpha\left(\mu^+ \circ \phi_\alpha^{-1}, q^{-1}\right) \cup U_\alpha\left(\mu^- \circ \phi_\alpha^{-1}, q^{-1}\right)\right) \cap M(K_\alpha)\right) \subset U. \tag{3.2}$$

By (3.1) and Proposition 2.5(a)(b), there exist sequences $\{\mu_n^+\}_{n\in\mathbb{N}}$, $\{\mu_n^-\}_{n\in\mathbb{N}}$ of CO-measures in $U \cap M_\alpha^{\mathrm{e}}(\Sigma_D)$ converging to μ^+, μ^- respectively such that $\mu_1^+ \neq \mu_1^-$, $\mu_n^+ \neq \mu_{n+1}^+$ and $\mu_n^- \neq \mu_{n+1}^-$ for all $n \in \mathbb{N}$, and the following two conditions hold:

(B1) For all $n \in \mathbb{N}$, $\int E_\alpha \circ \phi_\alpha \mathrm{d}\mu_n^+ > 1$ and $\int E_\alpha \circ \phi_\alpha \mathrm{d}\mu_n^- > 1$.

(B2) For all $n \in \mathbb{N}$,

$$\max\{d(\mu^+ \circ \phi_\alpha^{-1}, \mu_n^+ \circ \phi_\alpha^{-1}), d(\mu^- \circ \phi_\alpha^{-1}, \mu_n^- \circ \phi_\alpha^{-1})\} < (n+q)^{-1}.$$

Since E_α is continuous, the set

$$V_\alpha = \left\{ \nu \in M(\Sigma_\alpha)\colon \int E_\alpha \mathrm{d}\nu > 1 \right\}$$

is a convex open subset of $M(\Sigma_\alpha)$. For each $n \in \mathbb{N}$ we set

$$V_n^+ = U_\alpha\left(\mu^+ \circ \phi_\alpha^{-1}, (n+q)^{-1}\right) \cap V_\alpha \quad \text{and} \quad W_n^+ = \psi_\alpha^*(V_n^+ \cap M(K_\alpha)),$$
$$V_n^- = U_\alpha\left(\mu^- \circ \phi_\alpha^{-1}, (n+q)^{-1}\right) \cap V_\alpha \quad \text{and} \quad W_n^- = \psi_\alpha^*(V_n^- \cap M(K_\alpha)).$$

Note that V_n^+, V_n^- are convex open subsets of $M(\Sigma_\alpha)$, and they decrease as n increases. By (B1) and (B2), $\mu_n^+ \circ \phi_\alpha^{-1}$ belongs to $V_n^+ \cap M(K_\alpha)$ and $\mu_n^- \circ \phi_\alpha^{-1}$ belongs to $V_n^- \cap M(K_\alpha)$. By (3.2) we have

$$W_n^+ \cup W_n^- \subset U \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

If $\int E_\alpha \circ \phi_\alpha \mathrm{d}\mu^+ > 1$ (resp. $\int E_\alpha \circ \phi_\alpha \mathrm{d}\mu^+ = 1$), then $\mu^+ \in W_n^+$ (resp. $\mu^+ \notin W_n^+$) holds for all $n \in \mathbb{N}$. Analogous statements hold for μ^-. The next lemma asserts that W_n^+ (resp. W_n^-) approaches to μ^+ (resp. μ^-) as $n \to \infty$. See FIGURE 2 for a schematic picture.

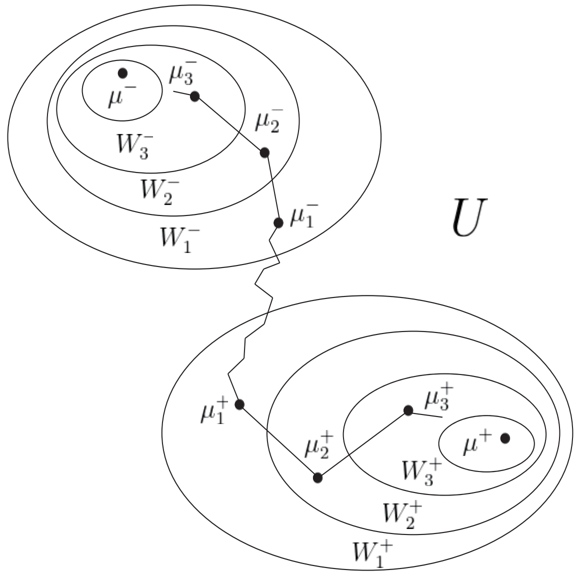

FIGURE 2. Each straight segment indicates the path that lies in $U \cap M_\alpha^{\mathrm{e}}(\Sigma_D)$ and joins the two CO-measures at its endpoints, which is obtained by applying Lemma 3.5 once. Their concatenation lies in $U \cap M_\alpha^{\mathrm{e}}(\Sigma_D)$ and joins μ^+, μ^-.

Lemma 3.6. *For any open subset Y of $M(\Sigma_D)$ that contains μ^+ (resp. μ^-), there exists $n \in \mathbb{N}$ such that $M^{\mathrm{e}}(\Sigma_D) \cap W_n^+ \subset Y$ (resp. $M^{\mathrm{e}}(\Sigma_D) \cap W_n^- \subset Y$).*

Proof. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $M^{\mathrm{e}}(\Sigma_D)$ such that $\rho_n \in M^{\mathrm{e}}(\Sigma_D) \cap W_n^+$ for all $n \in \mathbb{N}$. Let $\{\rho_{n(j)}\}_{j\in\mathbb{N}}$ be an arbitrary convergent subsequence and let ρ denote its limit measure. By the definitions of $V_{n(j)}^+$ and $W_{n(j)}^+$ we have $d(\mu^+\circ\phi_\alpha^{-1}, \rho_{n(j)}\circ\phi_\alpha^{-1}) < (n(j)+q)^{-1}$, and so $\rho_{n(j)} \circ \phi_\alpha^{-1} \to \mu^+ \circ \phi_\alpha^{-1}$. Meanwhile we have $\psi_\alpha^*(\rho_{n(j)} \circ \phi_\alpha^{-1}) = \rho_{n(j)} \to \rho$ and $\psi_\alpha^*(\mu_{n(j)}^+ \circ \phi_\alpha^{-1}) = \mu_{n(j)}^+ \to \mu^+$, and $\mu_{n(j)}^+ \circ \phi_\alpha^{-1} \to \mu^+ \circ \phi_\alpha^{-1}$ by (B2). The continuity of ψ_α^* at $\mu^+ \circ \phi_\alpha^{-1}$ yields $\rho = \mu^+$. Since $\{\rho_{n(j)}\}_{j\in\mathbb{N}}$ is an arbitrary convergent subsequence, the assertion of the lemma for μ^+ follows. A proof of the assertion of the lemma for μ^- is analogous. □

By Lemma 3.5, for each $n \in \mathbb{N}$ there is a homeomorphism $t \in [0,1] \mapsto \theta_{n,t}^+ \in V_n^+ \cap M^{\mathrm{e}}(\Sigma_\alpha)$ onto its image such that $\theta_{n,0}^+ = \mu_n^+ \circ \phi_\alpha^{-1}$, $\theta_{n,1}^+ = \mu_{n+1}^+ \circ \phi_\alpha^{-1}$, and $\theta_{n,t}^+$ is fully supported on Σ_α and is Bernoulli for all $t \in (0,1)$. Since $V_n^+ \subset V_\alpha$, we have $\int E_\alpha \mathrm{d}\theta_{n,t}^+ > 1$ for all $t \in [0,1]$. Lemma 2.3(a) gives $\theta_{n,t}^+ \in M(K_\alpha)$ for all $t \in [0,1]$. Hence, the measure $\mu_{n,t}^+ = \theta_{n,t}^+ \circ \psi_\alpha^{-1}$ is well-defined and belongs to $M_\alpha^{\mathrm{e}}(\Sigma_D)$ for all $t \in [0,1]$, and $t \in [0,1] \mapsto \mu_{n,t}^+ \in M_\alpha^{\mathrm{e}}(\Sigma_D)$ is a homeomorphism onto its image. This path lies in W_n^+ and joins μ_n^+, μ_{n+1}^+ by $\mu_{n,0}^+ = \mu_n^+$ and $\mu_{n,1}^+ = \mu_{n+1}^+$. By Lemma 3.4, $\mu_{n,t}^+$ is fully supported on Σ_D and is Bernoulli for all $t \in (0,1)$. In other words, $t \in [0,1] \mapsto \mu_{n,t}^+$ is a high complexity path. We repeat the same argument to obtain for each $n \in \mathbb{N}$ a high complexity path $t \in [0,1] \mapsto \mu_{n,t}^- \in M_\alpha^{\mathrm{e}}(\Sigma_D)$ that lies in W_n^- and joins μ_n^-, μ_{n+1}^- by $\mu_{n,0}^- = \mu_n^-$ and $\mu_{n,1}^- = \mu_{n+1}^-$.

Lemma 3.7. *There is a high complexity path that lies in $U \cap M^{\mathrm{e}}_{\alpha}(\Sigma_D)$ and joins μ_1^+, μ_1^-.*

Proof. Consider the set $L = \{t\mu_1^+ \circ \phi_\alpha^{-1} + (1-t)\mu_1^- \circ \phi_\alpha^{-1} \colon t \in [0,1]\}$. By (B1) and Lemma 2.3(a) we have $\mu_1^+ \circ \phi_\alpha^{-1}$, $\mu_1^- \circ \phi_\alpha^{-1} \in M(K_\alpha)$. Hence $L \subset M(K_\alpha)$ holds. Since $\psi_\alpha^*(\mu_1^+ \circ \phi_\alpha^{-1}) = \mu_1^+ \in U$, $\psi_\alpha^*(\mu_1^- \circ \phi_\alpha^{-1}) = \mu_1^- \in U$ and U is convex, we have $\psi_\alpha^*(L) = \{t\mu_1^+ + (1-t)\mu_1^- \colon t \in [0,1]\} \subset U$. Since ψ_α^* is continuous, $L \subset V_\alpha$ and L is a compact subset of $M(\Sigma_\alpha)$, there exist an integer $k \geq 3$ and convex open subsets $Q_1, \ldots, Q_k$ of $M(\Sigma_\alpha)$ such that:

(i) $(Q_i \cap Q_{i+1}) \cap L \neq \emptyset$ for $i = 1, \ldots, k-1$;
(ii) $L \subset \bigcup_{i=1}^k Q_i \subset V_\alpha$ and $\bigcup_{i=1}^k \psi_\alpha^*(Q_i \cap M(K_\alpha)) \subset U$;
(iii) $\mu_1^+ \circ \phi_\alpha^{-1} \in Q_1$ and $\mu_1^- \circ \phi_\alpha^{-1} \in Q_k$.

Since $M^{CO}(\Sigma_\alpha)$ is dense in $M(\Sigma_\alpha)$ by [34, Theorem 1], (i) implies $(Q_i \cap Q_{i+1}) \cap M^{CO}(\Sigma_\alpha) \neq \emptyset$ for $i = 1, \ldots, k-1$. For each $i \in \{1, \ldots, k-1\}$ we fix $\nu_i \in (Q_i \cap Q_{i+1}) \cap M^{CO}(\Sigma_\alpha)$ such that $\mu_1^+ \circ \phi_\alpha^{-1} \neq \nu_1$, $\nu_i \neq \nu_{i+1}$ for $i = 1, \ldots, k-2$ and $\nu_k \neq \mu_1^- \circ \phi_\alpha^{-1}$. By Lemma 3.5, the following statements hold:

- There is a high complexity path that lies in $Q_1 \cap M^{\mathrm{e}}(\Sigma_\alpha)$ and joins $\mu_1^+ \circ \phi_\alpha^{-1}$, ν_1.
- For each $i \in \{1, \ldots, k-2\}$, there is a high complexity path that lies in $Q_{i+1} \cap M^{\mathrm{e}}(\Sigma_\alpha)$ and joins ν_i, ν_{i+1}.
- There is a high complexity path that lies in $Q_k \cap M^{\mathrm{e}}(\Sigma_\alpha)$ and joins ν_k, $\mu_1^- \circ \phi_\alpha^{-1}$.

Concatenating all these paths yields a path that lies in $(\bigcup_{i=1}^k Q_i) \cap M^{\mathrm{e}}(\Sigma_\alpha)$ and joins $\mu_1^+ \circ \phi_\alpha^{-1}$, $\mu_1^- \circ \phi_\alpha^{-1}$. From (ii) and Lemma 2.3(a), this path can be transported to a high complexity path that lies in $U \cap M^{\mathrm{e}}_\alpha(\Sigma_D)$ and joins μ_1^+, μ_1^-. □

By Lemma 3.7, there is a high complexity path $t \in [0,1] \mapsto \mu_t^0 \in U \cap M^{\mathrm{e}}_\alpha(\Sigma_D)$ that joins μ_1^+, μ_1^- by $\mu_0^0 = \mu_1^+$ and $\mu_1^0 = \mu_1^-$. We define a map $t \in [0,1] \mapsto \mu_t \in U$ by

$$\mu_t = \begin{cases} \mu^+ & \text{for } t = 0, \\ \mu^+_{n, 2^{n+2}(t - 2^{-n-2})} & \text{for } t \in [2^{-n-2}, 2^{-n-1}],\ n \in \mathbb{N}, \\ \mu^0_{2(t-1/4)} & \text{for } t \in [1/4, 3/4], \\ \mu^-_{n, 2^{-n+2}(t-1+2^{n-1})} & \text{for } t \in [1-2^{n-1}, 1-2^{n-2}],\ n \in \mathbb{N}, \\ \mu^- & \text{for } t = 1. \end{cases}$$

The construction gives $\{\mu_t \colon t \in [2^{-n-2}, 2^{-n-1}]\} = \{\mu^+_{n,t} \colon t \in [0,1]\} \subset W_n^+$ and $\{\mu_t \colon t \in [1-2^{n-1}, 1-2^{n-2}]\} = \{\mu^-_{n,t} \colon t \in [0,1]\} \subset W_n^-$ for all $n \in \mathbb{N}$. So, Lemma 3.6 implies the continuity of $t \mapsto \mu_t$ at $t = 0$ and $t = 1$. Hence this path lies in $U \cap (M^{\mathrm{e}}_0(\Sigma_D) \cup M^{\mathrm{e}}_\alpha(\Sigma_D))$ and joins μ^+, μ^-. See FIGURE 2 for a schematic picture. Since it is piecewise homeomorphic, the set $\{t \in [0,1] \colon \mu_t = \mu\}$ is countable for all $\mu \in \{\mu_t \colon t \in [0,1]\}$. Therefore, this path is a high complexity path. This completes the proof of Proposition 3.1. □

3.4. **More on path connectedness.** In addition to Proposition 3.2, the following statement is of independent interest.

Proposition 3.8. *The spaces $M^{\mathrm{e}}_\alpha(\Sigma_D)$ and $M^{\mathrm{e}}_\beta(\Sigma_D)$ are path connected and locally path connected.*

Proof. Let $\gamma \in \{\alpha, \beta\}$. Slightly modifying the proof of Proposition 3.1, one can show that for any convex open subset U of $M(\Sigma_D)$ and for any pair of distinct measures in μ^+, $\mu^- \in U \cap M^{\mathrm{e}}_\gamma(\Sigma_D)$, there exists a high complexity path that lies in $U \cap M^{\mathrm{e}}_\gamma(\Sigma_D)$ and joins them. The path connectedness of $M^{\mathrm{e}}_\gamma(\Sigma_D)$ is a consequence of this with $U = M(\Sigma_D)$. Since any point in $M(\Sigma_D)$ has a neighborhood base consisting of convex open sets, the local path connectedness of $M^{\mathrm{e}}_\gamma(\Sigma_D)$ is also a consequence. □

4. Proofs of the main results

We are almost ready to complete the proofs of the main results of this paper. In Section 4.1 we collect a few ingredients on functional analysis. In Section 4.2 we complete the proof of Theorem A. In Section 4.3 we complete the proof of Theorem B.

4.1. **Functional analysis.** The proof of Theorem A(b) requires the same set of functional analytic ingredients in [33, Section 3.1]. Here we copy them for the reader's convenience.

For a Banach space V with a norm $\|\cdot\|$, let V^* denote the set of real-valued bounded linear functionals on V. For each $\mu \in V^*$ let $\|\mu\|$ denote the norm

$$\|\mu\| = \sup\{|\mu(f)| \colon f \in V,\ \|f\| = 1\}.$$

Let Λ, $\mu \in V^*$. We say:

- μ is *tangent* to Λ at $f \in V$ if $\mu(f) \leq \Lambda(f+g) - \Lambda(f)$ holds for all $g \in V$.
- μ is *bounded* by Λ if $\mu(f) \leq \Lambda(f)$ holds for all $f \in V$.
- Λ is *convex* if $\Lambda(tf + (1-t)g) \leq t\Lambda(f) + (1-t)\Lambda(g)$ holds for all f, $g \in V$ and $t \in [0,1]$.

Theorem 4.1 ([19], Theorem V.1.1)**.** *Let V be a Banach space and let $\Lambda \in V^*$ be convex and continuous. For any $\mu \in V^*$ that is bounded by Λ, any $f \in V$ and any $\varepsilon > 0$, there exist $\tilde\mu \in V^*$ and $\tilde f \in V$ such that $\tilde\mu$ is tangent to Λ at $\tilde f$ and*

$$\|\tilde\mu - \mu\| \leq \varepsilon \quad \textit{and} \quad \|\tilde f - f\| \leq \frac{1}{\varepsilon}(\Lambda(f) - \mu(f) + s),$$

where $s = \sup\{\mu(g) - \Lambda(g) \colon g \in V\} \leq 0$.

For a continuous map T of a compact metric space X, the functional Λ_T on $C(X)$ given by (1.1) is convex and continuous. The next lemma characterizes maximizing measures in terms of Λ_T. Recall that $C(X)^*$ can be identified with the set of (finite) signed Borel measures on X by Riesz's representation theorem.

Lemma 4.2 ([6], Lemma 2.3)**.** *Let T be a continuous map of a compact metric space X and let $f \in C(X)$. Then $\mu \in C(X)^*$ is tangent to Λ_T at f if and only if μ belongs to $M(X,T)$ and is f-maximizing.*

If T is a continuous map of X, then for any $\mu \in M(X,T)$ there exists a unique Borel probability measure b_μ on $M(X,T)$ such that $b_\mu(M^{\mathrm{e}}(X,T)) = 1$ and $\mu = \int_{M^{\mathrm{e}}(X,T)} \nu \mathrm{d}b_\mu(\nu)$. We call b_μ the barycenter of μ. Put

$$\mathrm{supp}(b_\mu) = \bigcap \{F \colon F \subset M(X,T), \text{ closed}, \ b_\mu(F) = 1\}.$$

Since $M(X,T)$ has a countable base, we have $b_\mu(\mathrm{supp}(b_\mu)) = 1$.

Lemma 4.3 ([33], Lemma 3.3)**.** *Let T be a continuous map of a compact metric space X.*

(a) *If there exists a constant $C \geq 0$ such that $h(\nu) \leq C$ for all $\nu \in \mathrm{supp}(b_\mu)$, then $h(\mu) \leq C$.*
(b) *Let $f \in C(X)$, $\mu \in M_{\max}(f)$. Then $\mathrm{supp}(b_\mu)$ is contained in $M_{\max}(f)$.*

The next lemma asserts that the barycenter map $\mu \mapsto b_\mu$ from $M(X,T)$ to the set of Borel probability measures on $M(X,T)$ is isometric.

Lemma 4.4 ([19], Corollary IV.4.2)**.** *Let T be a continuous map of a compact metric space X. For all μ, $\mu' \in M(X,T)$ we have*

$$\|b_\mu - b_{\mu'}\| = \|\mu - \mu'\|.$$

4.2. **Proof of Theorem A.** For each $n \in \mathbb{N}$, define

$$O_n = \left\{ \mu \in \overline{M^{\mathrm{e}}(\Sigma_D)} : 0 \leq h(\mu) < \frac{1}{n} \right\},$$

and

$$U_n = \{f \in C(\Sigma_D) \colon \overline{M^{\mathrm{e}}(\Sigma_D)} \cap M_{\max}(f) \subset O_n\}.$$

Clearly, the set $\mathscr{R} = \{f \in C(\Sigma_D) \colon h(\mu) = 0 \text{ for all } \mu \in M_{\max}(f)\}$ is contained in $\bigcap_{n=1}^\infty U_n$. Conversely, let $f \in \bigcap_{n=1}^\infty U_n$. Any ergodic measure in $M_{\max}(f)$ has zero entropy, and by Lemma 4.3, any non-ergodic measure in $M_{\max}(f)$ has zero entropy too. Hence we obtain $\mathscr{R} = \bigcap_{n=1}^\infty U_n$.

By the upper semicontinuity of the entropy function, O_n is an open subset of $\overline{M^{\mathrm{e}}(\Sigma_D)}$. CO-measures have zero entropy, and they are dense in $\overline{M^{\mathrm{e}}(\Sigma_D)}$ by Proposition 2.5(a). It follows that O_n is a dense subset of $\overline{M^{\mathrm{e}}(\Sigma_D)}$. By the result of Morris [29, Theorem 1.1], U_n is an open and dense subset of $C(\Sigma_D)$. Therefore, $\mathscr{R}$ is dense G_δ as required in Theorem A(a).

To prove Theorem A(b), let $f \in C(\Sigma_D)$. By Proposition 3.1, there is a high complexity path $t \in [0,1] \mapsto \mu_t \in M^{\mathrm{e}}(\Sigma_D)$ such that $\mu_0 \in M_{\max}(f)$. Let $\varepsilon \in (0, 1/2)$ and put

$$R = \left\{ \nu \in M(\Sigma_D) \colon \int f \mathrm{d}\nu \geq \Lambda_\sigma(f) - \varepsilon^2 \right\}.$$

Let m denote the Lebesgue measure on $[0,1]$, and define a Borel probability measure $\hat{m}$ on $M^{\mathrm{e}}(\Sigma_D)$ by $\hat{m}(\cdot) = m\{t \in [0,1] \colon \mu_t \in \cdot\}$. Since $f \in C(\Sigma_D)$, we have $\hat{m}(R \cap M^{\mathrm{e}}(\Sigma_D)) > 0$. Let $\hat{m}_R$ denote the normalized restriction of $\hat{m}$ to $R \cap M^{\mathrm{e}}(\Sigma_D)$, and put $\mu = \int_{M^{\mathrm{e}}(\Sigma_D)} \nu \mathrm{d}\hat{m}_R(\nu)$. Then μ belongs to R and is bounded by Λ_σ as an element of $C(\Sigma_D)^*$. Note that $b_\mu = \hat{m}_R$. By Theorem 4.1, there exist $\tilde{f} \in C(\Sigma_D)$ and $\tilde{\mu} \in C(\Sigma_D)^*$ such that $\tilde{\mu}$ is tangent to Λ_σ at $\tilde{f}$, and

$$\|\tilde{\mu} - \mu\| \leq \varepsilon \ \text{ and } \ \|\tilde{f} - f\|_{C^0} \leq \frac{1}{\varepsilon}\left(\Lambda_\sigma(f) - \int f \mathrm{d}\mu\right) \leq \varepsilon. \tag{4.1}$$

By Lemma 4.2, $\tilde{\mu}$ is shift-invariant and $\tilde{f}$-maximizing.

Take an open subset U of $M(\Sigma_D)$ such that $\mathrm{supp}(b_{\tilde{\mu}}) \subset U$ and $b_\mu(U \setminus \mathrm{supp}(b_{\tilde{\mu}})) < \varepsilon$. Since $M^{\mathrm{e}}(\Sigma_D)$ is a metric space, it is a normal space. By Urysohn's lemma, there exists $g \in C(M(\Sigma_D))$ such that $\|g\|_{C^0} = 1$, $g \equiv 0$ on $M(\Sigma_D) \setminus U$ and $g \equiv 1$ on $\mathrm{supp}(b_{\tilde{\mu}})$. We have

$$\begin{aligned} b_\mu(\mathrm{supp}(b_{\tilde{\mu}})) &> b_\mu(U) - \varepsilon > b_\mu(g) - \varepsilon \geq b_{\tilde{\mu}}(g) - 2\varepsilon \\ &\geq b_{\tilde{\mu}}(\mathrm{supp}(b_{\tilde{\mu}})) - 2\varepsilon = 1 - 2\varepsilon > 0. \end{aligned}$$

To deduce the third inequality, we have used $\|b_{\tilde{\mu}} - b_\mu\| = \|\tilde{\mu} - \mu\| \leq \varepsilon$ from Lemma 4.4 and the first inequality in (4.1). Since b_μ is non-atomic from the property (A2), it follows that the set $\{\mu_t \colon t \in [0,1],\ \mu_t \in \mathrm{supp}(b_{\tilde{\mu}})\}$ contains uncountably many elements, which belong to $M_{\max}(\tilde{f})$ by Lemma 4.3. Since $f \in C(\Sigma_D)$ and $\varepsilon \in (0, 1/2)$ are arbitrary, the proof of Theorem A(b) is complete. □

4.3. **Proof of Theorem B.** Since $M^{\mathrm{e}}(\Sigma_D) = M^{\mathrm{e}}_\alpha(\Sigma_D) \cup M^{\mathrm{e}}_\beta(\Sigma_D) \cup M^{\mathrm{e}}_0(\Sigma_D)$, the path connectedness of $M^{\mathrm{e}}(\Sigma_D)$ follows from Proposition 3.2. Let U be a convex open subset of $M(\Sigma_D)$. Since $M^{\mathrm{e}}(\Sigma_D)$ is dense in $M(\Sigma_D)$, we have $U \cap M^{\mathrm{e}}(\Sigma_D) \neq \emptyset$. By Proposition 3.1, for each $\gamma \in \{\alpha, \beta\}$ the set $U \cap (M^{\mathrm{e}}_0(\Sigma_D) \cup M^{\mathrm{e}}_\gamma(\Sigma_D))$ is path connected unless empty. It follows that $U \cap M^{\mathrm{e}}(\Sigma_D)$ is path connected. Since any point in $M(\Sigma_D)$ has a neighborhood base consisting of convex open sets and U is an arbitrary convex open subset of $M(\Sigma_D)$, we have verified that $M^{\mathrm{e}}(\Sigma_D)$ is locally path connected. The proof of Theorem B is complete. □

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