ON THE DENSITY OF PERIODIC MEASURES FOR PIECEWISE MONOTONIC MAPS AND THEIR CODING SPACES

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ABSTRACT. We prove that for all transitive piecewise monotonic maps, the following conditions are equivalent:
(1) All ergodic measures on $[0,1]$ are approximated by periodic ones;
(2) All invariant measures on coding spaces are approximated by periodic ones;
(3) All ergodic measures on coding spaces are approximated by periodic ones.
If we further assume that the map is piecewise increasing and either right or left continuous, then the following condition is also equivalent to (1)-(3).
(4) All invariant measures on $[0,1]$ are approximated by periodic ones.
We also construct an example of a piecewise decreasing and right continuous map which satisfies (1)-(3), but does not satisfy (4).

1. INTRODUCTION

In this paper, we consider piecewise monotonic maps on the unit interval $[0,1]$. We say that a measurable map $T\colon [0,1] \to [0,1]$ is *piecewise monotonic* if there exist integer $k > 1$ and $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $T|_{(a_{j-1},a_j)}$ is strictly monotonic and continuous for each $1 \leq j \leq k$. If $T|_{(a_{j-1},a_j)}$ is increasing (resp. decreasing) for $1 \leq j \leq k$, then we call T a *piecewise increasing* (resp. *decreasing*) map. Throughout of this paper, we assume two further conditions:

- The map T is *transitive* i.e. there exists a point $x \in [0,1]$ whose forward orbit $\{T^n(x) : n \geq 0\}$ is dense in $[0,1]$.
- The topological entropy $h_{\text{top}}(T)$ of T is positive (see [8, §3] and [5, §9] for the definition and basic properties of the topological entropy).

Dynamical properties of piecewise monotonic maps are very important from viewpoints of ergodic theory and number theory, and therefore they have been studied by many authors.

For a piecewise monotonic map $T\colon [0,1] \to [0,1]$, one usually consider its coding space (Σ_T^+, σ), which is a subshift consisting of all kneading sequences induced by T. To be more precise, let $\Sigma_k^+ := \{1,\dots,k\}^{\mathbb{N}}$ be a product space with the product topology of the discrete topology of $\{1,\dots,k\}$ and $\sigma\colon \Sigma_k^+ \to \Sigma_k^+$ be a shift map defined as usual. Let $X_T := \bigcap_{n\geq 0} T^{-n}(\bigcup_{j=1}^k (a_{j-1},a_j))$ and define the *coding map* $\mathcal{I}\colon X_T \to \Sigma_k^+$ by $(\mathcal{I}(x))_i = j$ if and only if $T^{i-1}(x) \in (a_{j-1},a_j)$. Denote the closure of $\mathcal{I}(X_T)$ by Σ_T^+ and call the system (Σ_T^+, σ) the *coding space* of $([0,1],T)$. Then by the

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transitivity of T, one can easily prove that $\mathcal{I}$ is injective (see the appendix of this paper). This together with [8, Theorem 2.5] imply the following:

- $\mathcal{I}\colon X_T \to \mathcal{I}(X_T)$ is a homeomorphism and $\mathcal{I} \circ T(x) = \sigma \circ \mathcal{I}(x)$ for any $x \in X_T$.
- Both $[0,1] \setminus X_T$ and $\Sigma_T^+ \setminus \mathcal{I}(X_T)$ are countable and contain at most finite periodic points.

Roughly speaking, the above states that $([0,1],T)$ is topologically conjugate to (Σ_T^+, σ) except the set of (only countably many) points whose forward orbit contains at least one discontinuity point. For dynamical properties of piecewise monotonic maps, the behavior at the discontinuity points is often neglected and hence the coding space (Σ_T^+, σ) is studied instead of $([0,1],T)$ frequently. However, if we investigate piecewise monotonic maps from viewpoint of number theory, the behavior at the discontinuity points is also important and not negligible in many cases. Hence it is worth to make clear the difference between dynamical properties of $([0,1],T)$ and (Σ_T^+, σ). In this paper, we investigate the difference between the density of periodic measures for piecewise monotonic maps and their coding spaces.

For a metric space X and a Borel measurable map $f\colon X \to X$, denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X with the weak*-topology, by $\mathcal{M}_f(X) \subset \mathcal{M}(X)$ the set of f-invariant ones, and by $\mathcal{M}_f^e(X) \subset \mathcal{M}_f(X)$ the set of ergodic ones. We say that $\mu \in \mathcal{M}(X)$ is a *periodic measure* if there is a periodic point $x \in X$ such that $\mu = \mathcal{P}_f(x) := (\sum_{j=0}^{n-1} \delta_{f^j(x)})/n$, where n is the minimal period of x and δ_y is the Dirac mass at the point $y \in X$. Then it is clear that $\mathcal{P}_f(x) \in \mathcal{M}_f^e(X)$. We let $\mathcal{M}_f^p(X) \subset \mathcal{M}_f^e(X)$ the set of all periodic measures on X.

The property "$\mathcal{M}_f^p(X)$ is dense in $\mathcal{M}_f(X)$ (or in $\mathcal{M}_f^e(X)$)" is important to investigate the space of invariant measures $\mathcal{M}_f(X)$ and hence the density of periodic measures have been studied for several dynamical systems such as piecewise monotonic maps ([14, 15, 16, 24]) and other systems ([1, 3, 4, 9, 10, 17, 20, 22, 23]). Moreover, the present author et al. recently found that the density of periodic measures plays a key role to prove several thermodynamic properties for piecewise monotonic maps such as the thermodynamic dichotomy for irregular sets and large deviations ([7, 19]). In [19], it is proved that for transitive piecewise monotonic maps, the set of points for which the Birkhoff average of a continuous function does not exist is either empty or have full topological entropy under the condition that every ergodic measure is approximated by periodic ones. To obtain large deviations results, we require the stronger condition; every invariant measure is approximated by periodic ones (see the proof of [7, Theorem B]). The first main result of this paper implies that these two conditions are actually equivalent for transitive piecewise monotonic maps.

Theorem A. *Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map with $h_{\mathrm{top}}(T) > 0$ and (Σ_T^+, σ) be its coding space. Then the following are equivalent:*

(a) $\mathcal{M}_T^p(X_T)$ *is dense in* $\mathcal{M}_T(X_T)$*;*
(b) $\mathcal{M}_T^p([0,1])$ *is dense in* $\mathcal{M}_T^e([0,1])$*;*
(c) $\mathcal{M}_\sigma^p(\Sigma_T^+)$ *is dense in* $\mathcal{M}_\sigma(\Sigma_T^+)$*;*
(d) $\mathcal{M}_\sigma^p(\Sigma_T^+)$ *is dense in* $\mathcal{M}_\sigma^e(\Sigma_T^+)$*.*

As we said before, the density of periodic measures for piecewise montonic maps is very important and has been studied by several authors. In particular, it is proved in [14, 16, 23] that the condition (c) (and hence conditions (a)-(d)) holds for the following important classes of transitive piecewise monotonic maps with positive topological entropy:

- The map T has the specification property (see [6, §1] for the definition).
- The map T is a *monotonic mod one transformation*, i.e. there exists a strictly increasing and continuous function $f\colon [0,1] \to \mathbb{R}$ such that $T(x) = f(x) \bmod 1$.
- The map T has *two* intervals of monotonicity, i.e. there is a point $0 < a < 1$ such that both $T|_{(0,a)}$ and $T|_{(a,1)}$ are strictly monotonic and continuous.

In this paper, we give an example of piecewise monotonic maps which does not belong to any of above classes, but satisfies conditions (a)-(d).

We note that the condition (a) in Theorem A cannot be replaced the following condition in general:
(e) $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T([0,1])$.
Hence it is natural to ask for which transitive piecewise monotonic maps, conditions (a)-(e) are equivalent. It is clear that this holds if a_j is not periodic for each $1 \leq j \leq k$ (indeed, one can easily see that $\mathcal{M}_T([0,1]) = \mathcal{M}_T(X_T)$ for such maps). The second aim of this paper is to give a nontrivial, and simple sufficient condition for the equivalence of (a)-(e).

Theorem B. *Let* $T\colon [0,1] \to [0,1]$ *be a transitive piecewise monotonic map with* $h_{\mathrm{top}}(T) > 0$ *and* (Σ_T^+, σ) *be its coding space. Assume that* T *is piecewise increasing and either right or left continuous. Then conditions (a)-(e) are equivalent.*

We remark that the assumption "piecewise increasing" in Theorem B cannot be removed. Indeed, we can construct an example of a piecewise decreasing and right continuous map which satisfies (a)-(d), but does not satisfy (e) (see Example 3.1).

This paper is organized as follows: In §2, we establish our definitions, recall several basic facts, and prove lemmas which play key roles to prove Theorems A and B. In §3, we give a proof of Theorem A and also prove Theorem B in §4. In §5, we give a new example of piecewise monotonic maps satisfying conditions (a)-(e). In the appendix of this paper §6, we prove that the transitivity of a piecewise monotonic map implies the injectivity of the coding map.

2. Preliminaries

Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map and (Σ_T^+, σ) be its coding space. Recall that the coding map $\mathcal{I}\colon X_T \to \Sigma_T^+$ is injective by the transitivity of T (see §6). Hence one can see that the set

$$\bigcap_{n\geq 0} \mathrm{cl}(T^{-n}(a_{\omega_{n+1}-1}, a_{\omega_{n+1}})) \tag{2.1}$$

is a unit set for any $(\omega_i)_{i\in\mathbb{N}} \in \Sigma_T^+$. Here $\mathrm{cl}(A)$ denotes the closure of the set A. Then we define a map $\Phi\colon \Sigma_T^+ \to [0,1]$ by $\Phi((\omega_i)_{i\in\mathbb{N}}) = y$, where y is a unique element of the set (2.1). The following lemma is well-known and hence we omit the proof (see [8, Theorem 2.15] for instance):

Lemma 2.1. (1) The map Φ is continuous, surjective and $\Phi\circ\sigma(\omega) = T\circ\Phi(\omega)$ for any $\omega \in \mathcal{I}(X_T)$.
(2) We have $\Phi(\mathcal{I}(X_T)) = X_T$ and the restriction map $\Phi\colon \mathcal{I}(X_T) \to X_T$ is a inverse map of $\mathcal{I}\colon X_T \to \mathcal{I}(X_T)$.

In what follows we will construct Hofbauer's Markov Diagram, which is a countable oriented graph with subsets of Σ_T^+ as vertices. For each $1 \leq j \leq k$, we set $[j] := \{(\omega_i)_{i\in\mathbb{N}} \in \Sigma_T^+ : \omega_1 = j\}$.

Definition 2.2. Let $C \subset \Sigma_T^+$ be a closed subset with $C \subset [j]$ for some $1 \leq j \leq k$. We say that a non-empty closed subset $D \subset \Sigma_T^+$ is a *successor* of C if $D = [l] \cap \sigma(C)$ for some $1 \leq l \leq k$.

Now we define a set $\mathcal{D}_T$ of vertices by induction. First, we set $\mathcal{D}_0 := \{[1], \ldots, [k]\}$. If $\mathcal{D}_n$ is defined for $n \geq 0$, then we set

$$\mathcal{D}_{n+1} := \mathcal{D}_n \cup \{D : \text{there exists } C \in \mathcal{D}_n \text{ so that } D \text{ is a successor of } C\}.$$

We note that $\mathcal{D}_n$ is a finite set for each $n \geq 0$ since the number of successors of any closed subset of Σ_T^+ is at most k by the definition. Finally, we set

$$\mathcal{D}_T := \bigcup_{n\geq 0} \mathcal{D}_n.$$

To get the oriented graph, we insert an arrow from every $C \in \mathcal{D}_T$ to all its successors. We write $C \to D$ to denote that D is a successor of C. We call the oriented graph $(\mathcal{D}_T, \to)$ *Hofbauer's Markov Diagram* of T. Let $\mathcal{C} \subset \mathcal{D}_T$. We say that a sequence $C_1 \cdots C_n$ is a *finite path* in $\mathcal{C}$ if $C_1, \ldots, C_n \in \mathcal{C}$ and $C_i \to C_{i+1}$ for $1 \leq i \leq n-1$. A *one-sided infinite path* in $\mathcal{C}$ is defined analogously and denote by $\Sigma_{\mathcal{C}}^+$ the set of all one-sided infinite paths in $\mathcal{C}$. We say that $\mathcal{C}$ is *irreducible* if for any $C, D \in \mathcal{C}$, we can find a finite path $C_1 C_2 \cdots C_n$ in $\mathcal{C}$ with $C_1 = C$ and $C_n = D$ and if every subset of $\mathcal{D}_T$, which contains $\mathcal{C}$ strictly, does not have this property. Finally, we define $\Psi\colon \Sigma_{\mathcal{C}}^+ \to \Sigma_k^+$ by

$$\Psi((C_i)_{i\in\mathbb{N}}) := (\omega_i)_{i\in\mathbb{N}} \text{ for } (C_i)_{i\in\mathbb{N}} \in \Sigma_{\mathcal{C}}^+,$$

where $1 \le \omega_i \le k$ is the unique integer such that $C_i \subset [\omega_i]$ holds for each $i \in \mathbb{N}$.

Lemma 2.3. Let $\mu_1^+, \mu_2^+ \in \mathcal{M}_\sigma^p(\Sigma_T^+)$. Then there is a sequence $\{\nu_n^+\}_{n\ge 1} \subset \mathcal{M}_\sigma^p(\Sigma_T^+)$ such that $\nu_n^+ \to (\mu_1^+ + \mu_2^+)/2$ as $n \to \infty$.

Proof. For each $j = 1, 2$, we take a periodic point $x^j \in \Sigma_T^+$ in the support of μ_j^+, and denote by m_j the minimal period of x^j. Since T is transitive and $h_{\rm top}(T) > 0$, it follows from [13, Theorem 11] that there is an irreducible subset $\mathcal{C} \subset \mathcal{D}_T$ such that $\Sigma_T^+ = \Psi(\Sigma_\mathcal{C}^+)$ holds. By [13, Theorem 8], for each $j = 1, 2$, we can find an integer $p_j \in \{1, 2\}$ and $(C_i^j)_{i\in\mathbb{N}} \in \Sigma_\mathcal{C}^+$ satisfying the following properties:

- $C_i^j = C_{i+p_jm_j}^j$ for any $i \ge 1$.
- $\Psi((C_i^j)_{i\in\mathbb{N}}) = x^j$.

Since $\mathcal{C}$ is irreducible, we can find two integers s, t and finite paths $D_1 \cdots D_s$ and $E_1 \cdots E_t$ in $\mathcal{C}$ such that $C_{p_1m_1}^1 \to D_1$, $D_s \to C_1^2$, $C_{p_2m_2}^2 \to E_1$ and $E_t \to C_1^1$ hold. For each $n \ge 1$ and $j = 1, 2$, we denote $l_{n,j} := np_1p_2m_j$, and define $(F_i^n)_{i\in\mathbb{N}} \in \mathcal{C}^\mathbb{N}$ by

$$F_i^n = \begin{cases} C_i^1 & (1 \le i \le l_{n,1}), \\ D_{i-l_{n,1}} & (l_{n,1}+1 \le i \le l_{n,1}+s), \\ C_{i-(l_{n,1}+s)}^2 & (l_{n,1}+s+1 \le i \le l_{n,1}+l_{n,2}+s), \\ E_{i-(l_{n,1}+l_{n,2}+s)} & (l_{n,1}+l_{n,2}+s+1 \le i \le l_{n,1}+l_{n,2}+s+t), \end{cases}$$

and $F_{i+q(l_{n,1}+l_{n,2}+s+t)}^n = F_i^n$ for any $q \ge 1$ and $1 \le i \le l_{n,1}+l_{n,2}+s+t$. By construction, we have $(F_i^n)_{i\in\mathbb{N}} \in \Sigma_\mathcal{C}^+$ and hence we can define $z^n := \Psi((F_i^n)_{i\in\mathbb{N}})$. Then it is easy to see that z^n is periodic and $\mathcal{P}_\sigma(z^n)$ converges to $(\mu_1^+ + \mu_2^+)/2$, which prove the lemma. □

Lemma 2.3 together with an easy calculation imply the following:

Lemma 2.4. $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in the set of all finite convex combination of periodic measures.

We define a push forward map $\Phi_*\colon \mathcal{M}(\Sigma_T^+) \to \mathcal{M}([0,1])$ by $\Phi_*(\mu) := \mu \circ \Phi^{-1}$ for $\mu \in \mathcal{M}(\Sigma_T^+)$. Note that Φ_* is continuous by the continuity of Φ.

Lemma 2.5. Assume that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. Then $\mathcal{M}_\sigma^p(\mathcal{I}(X_T))$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$.

Proof. Let $\mu^+ \in \mathcal{M}_\sigma(\Sigma_T^+)$ and $\mathcal{U} \subset \mathcal{M}(\Sigma_T^+)$ be an open neighborhood of μ^+. Since T is transitive and $h_{\rm top}(T) > 0$, it follows from [12, Theorem 4] that there exists a unique (and hence ergodic) measure m of maximal entropy of $([0,1], T)$. Then it follows from [8, Theorem 5.6 and Proposition 5.7] that there exsits an ergodic measure m^+ on Σ_T^+ such that $h(m^+) = h(m) = h_{\rm top}(T) > 0$. Here $h(\mu^+)$ and $h(\mu)$ denotes the metric entropy of μ^+ and μ respectively (see [8, §3] for the definition). Note that μ^+ is non-atomic (otherwise, $h(\mu^+) = 0$ by the ergodicity of μ^+, which is a contradiction).

Hence we have $m^+(\mathcal{I}(X_T)) = 1$ since $\Sigma_T^+ \setminus \mathcal{I}(X_T)$ is countable. Choose $\epsilon > 0$ so small that $\nu^+ := (1-\epsilon)\mu^+ + \epsilon m^+ \in \mathcal{U}$. Since $\mathcal{M}_\sigma^p(\Sigma_T^+) \setminus \mathcal{M}_\sigma^p(\mathcal{I}(X_T))$ is finite and does not contain ν^+, we can find an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of ν^+ such that $\mathcal{U}' \cap \mathcal{M}_\sigma^p(\Sigma_T^+) \subset \mathcal{M}_\sigma^p(\mathcal{I}(X_T))$. Since $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$, there is a periodic measure $\rho^+ \in \mathcal{U}'$. Hence we have $\rho^+ \in \mathcal{M}_\sigma^p(\mathcal{I}(X_T)) \cap \mathcal{U}$, which proves the lemma. □

Lemma 2.6. Assume that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. Then we have $\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \mathrm{cl}(\mathcal{M}_T(X_T))$.

Proof. Since the restriction map $\Phi\colon \mathcal{I}(X_T) \to X_T$ is a topological conjugacy map by Lemma 2.1, we have $\Phi_*(\mathcal{M}_\sigma(\mathcal{I}(X_T))) = \mathcal{M}_T(X_T)$. Hence by Lemma 2.5 and the continuity of Φ_*, we have

$$\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \Phi_*(\mathrm{cl}(\mathcal{M}_\sigma(\mathcal{I}(X_T)))) = \mathrm{cl}(\Phi_*(\mathcal{M}_\sigma(\mathcal{I}(X_T)))) = \mathrm{cl}(\mathcal{M}_T(X_T)),$$

which proves the lemma. □

Lemma 2.7. Assume that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. Then $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T([0,1])$ if and only if $\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \mathcal{M}_T([0,1])$.

Proof. Suppose that $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T([0,1])$. Then we can prove $\mathcal{M}_T^p(X_T)$ is dense in $\mathcal{M}_T([0,1])$ in a similar way to the proof of Lemma 2.5, which implies that $\mathrm{cl}(\mathcal{M}_T(X_T)) = \mathcal{M}_T([0,1])$. Hence by Lemma 2.6, we have $\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \mathcal{M}_T([0,1])$.

Next, we assume that $\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \mathcal{M}_T([0,1])$. Then for any $\mu \in \mathcal{M}_T([0,1])$, we can find $\mu^+ \in \mathcal{M}_\sigma(\Sigma_T^+)$ such that $\Phi_*(\mu^+) = \mu$. Since $\mathcal{M}_\sigma^p(\mathcal{I}(X_T))$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$ by Lemma 2.5, we can find a sequence $\{\mu_n^+\}_{n\geq 1} \subset \mathcal{M}_\sigma^p(\mathcal{I}(X_T))$ such that $\lim_{n\to\infty} \mu_n^+ = \mu^+$. Then it is clear that $\{\Phi_*(\mu_n^+)\}_{n\geq 1} \subset \mathcal{M}_T^p([0,1])$ and hence by the continuity of Φ_*, we have $\lim_{n\to\infty} \Phi_*(\mu_n^+) = \mu$. □

3. Proof of Theorem A

The aim of this section is to give a proof of Theorem A. Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map with $h_{\mathrm{top}}(T) > 0$. First, we show (a)⇒(b). Suppose that $\mathcal{M}_T^p(X_T)$ is dense in $\mathcal{M}_T(X_T)$ and let $\mu \in \mathcal{M}_T^e([0,1])$. Since X_T is T-invariant and μ is ergodic, we have either $\mu(X_T) = 1$ or $\mu([0,1] \setminus X_T) = 1$. In the former case, μ can be approximated by periodic measures by the assumption (a). In the latter case, $\mu \in \mathcal{M}_T^p([0,1])$ holds since $[0,1] \setminus X_T$ is countable. Hence we have (b). Now we prove (b)⇒(d). Since there are only finitely many periodic points in $[0,1] \setminus X_T$, (b) implies that $\mathcal{M}_T^p(X_T)$ is dense in $\mathcal{M}_T^e(X_T)$. Recall that (X_T, T) is topologically conjugate to $(\mathcal{I}(X_T), \sigma)$. Hence we have $\mathcal{M}_\sigma^p(\mathcal{I}(X_T))$ is dense in $\mathcal{M}_\sigma^e(\mathcal{I}(X_T))$. Since $\Sigma_T^+ \setminus \mathcal{I}(X_T)$ is countable, we have $\mathcal{M}_\sigma^e(\Sigma_T^+) \setminus \mathcal{M}_\sigma^e(\mathcal{I}(X_T)) \subset \mathcal{M}_\sigma^p(\Sigma_T^+)$. Therefore we have (d). A proof of the implication (c)⇒(a) is given by a similar way to that of (b)⇒(d).

Finally, we show (d)⇒(c). It follows from Ergodic Decomposition Theorem that the set of all convex combination of ergodic measures is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. This together with the assumption (d) imply that the set of all convex combination of periodic measures is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. Hence (c) follows from Lemma 2.4. This completes the proof of Theorem A.

Example 3.1. In what follows we will give an example of a transitive piecewise decreasing map $T\colon [0,1] \to [0,1]$ satisfying the following properties:

- T is right continuous;
- T satisfies conditions (a)-(d), but does not satisfy the condition (e).

Let $\beta = 1.7548\cdots$ be a unique real root of the equation $X^3-2X^2+X-1 = 0$ and define $T\colon [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} -\beta x + 1 & (0 \le x < \frac{1}{\beta}) \\ -\beta x + 2 & (\frac{1}{\beta} \le x \le 1). \end{cases}$$

In Figure 1, we sketch the graph of T. Then by an easy calculation, we

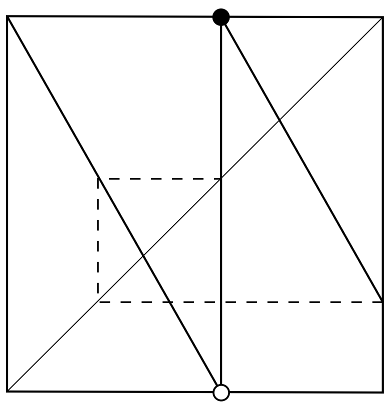

FIGURE 1. A graph of T.

have the following:

(3.1) $T^3(1) = 1$ and $T(1/(1+\beta)) = 1/(1+\beta)$;

(3.2) There exists an open neighborhood U of the set $\{1, T(1), T^2(1)\}$ such that for any $x \in U \setminus \{1, T(1), T^2(1)\}$, we can find an integer $1 \le i \le 5$ such that $T^i(x) \notin U$.

First, we prove that T satisfies conditions (a)-(d). Since $\beta > (1+\sqrt{5})/2$, it follows from [18, Theorem 2.2] that T is transitive. Moreover, it follows from [5, Theorem 9.4.1] and [8, Proposition 3.6] that $h_{\mathrm{top}}(T) = \log\beta > 0$. Hence by [16, Theorem 2], T satisfies the condition (c). Again by the transitivity of T, $h_{\mathrm{top}}(T) > 0$ and Theorem A, T also satisfies conditions (a), (b) and (d).

Finally, we will show that T does not satisfy the condition (e). We set

$$\mu := \frac{9}{10}\mathcal{P}_T(1) + \frac{1}{10}\mathcal{P}_T\left(\frac{1}{1+\beta}\right).$$

Let U be an open neighborhood of $\{1, T(1), T^2(1)\}$ as in (3.2) and take any periodic point $x \in [0,1]$ with $\mathcal{P}_T(x) \neq \mathcal{P}_T(1)$. Then it follows from (3.2) that $\mathcal{P}_T(x)(U) \leq 5/6 < 9/10 = \mu(U)$ holds, which implies that μ cannot be approximated by periodic measures.

4. Proof of Theorem B

In this section, we prove Theorem B. Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map with $h_{\mathrm{top}}(T) > 0$ and assume that T is piecewise increasing and either right or left continuous. It is sufficient to show the implication (c)⇒(e). First, we assume that T is right continuous. We extend the map $\mathcal{I}$ to the map $\mathcal{J}$ on $[0,1]$ by $(\mathcal{J}(x))_i = j$ if $T^{i-1}(x) \in I_j$. Here we set $I_j := [a_{j-1}, a_j)$ for $1 \leq j \leq k-1$ and $I_k := [a_{k-1}, 1]$. Since T is piecewise increasing and right continuous, we can prove that $\mathcal{J}$ is also right continuous in a similar way to the proof of [8, Lemma 2.4] (see also [8, §7.2]). Denote by $S(T)$ the set of integers $0 \leq j \leq k$ so that a_j is periodic.

Lemma 4.1. For any $j \in S(T)$, we have $\mathcal{J}(a_j) \in \Sigma_T^+$.

Proof. Since $\mathcal{J}$ is right continuous, for any $0 \leq j \leq k-1$, we have

$$\mathcal{J}(a_j) = \lim_{x \to a_j+0, x \in X_T} \mathcal{J}(x) = \lim_{x \to a_j+0, x \in X_T} \mathcal{I}(x) \in \Sigma_T^+.$$

Assume that $a_k = 1$ is periodic. Since T is piecewise increasing and right continuous, we have $T(x) \neq 1$ for any $0 \leq x < 1$, which implies that a_k is a fixed point. Hence it is easy to see that

$$\mathcal{J}(a_k) = \lim_{x \to 1-0, x \in X_T} \mathcal{I}(x) = kkk\cdots \in \Sigma_T^+.$$

This proves the lemma. □

Now, suppose that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$.

Lemma 4.2. We have $\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \mathcal{M}_T([0,1])$.

Proof. First, we note that $\mathcal{M}_T^e([0,1]) \setminus \mathcal{M}_T^e(X_T) = \{\mathcal{P}_T(a_j) : j \in S(T)\}$. Hence for any $\mu \in \mathcal{M}_T([0,1])$, we can find $0 \leq c \leq 1$, $\{c_j\}_{j \in S(T)} \subset [0,1]$ and $\nu \in \mathcal{M}_T(X_T)$ such that $\mu = c\nu + \sum_{j \in S(T)} c_j \mathcal{P}_T(a_j)$ and $c + \sum_{j \in S(T)} c_j = 1$. By Lemma 4.1, for any $j \in S(T)$, we have $\mathcal{J}(a_j) \in \Sigma_T^+$. Hence by the definition of Φ, $\Phi_*(\mathcal{P}_\sigma(\mathcal{J}(a_j))) = \mathcal{P}_T(a_j)$ holds. Therefore, we have

$$\Phi_*\left(c\nu \circ \mathcal{I}^{-1} + \sum_{j \in S(T)} c_j \mathcal{P}_\sigma(\mathcal{J}(a_j))\right) = \mu,$$

which proves the lemma. □

Then, (e) directly follows from Lemmas 2.7 and 4.2. A proof of the case that T is left continuous is given by a similar way. Theorem B is proved.

5. Piecewise monotonic maps satisfying (a)-(e)

The aim of this section is to give a new example of piecewise monotnic maps satisfying conditions (a)-(e). As we said in §1, it is known that conditions (a)-(d) hold for the following three important classes of transitive piecewise monotonic maps with positive topological entropy:

(I) The map T has the specification property.
(II) The map T is a monotonic mod one transformation.
(III) The map T has two intervals of monotonicity.

In what follows we will construct an example of piecewise monotonic maps, which does not belong to any of (I)-(III), but satisfies conditions (a)-(e). For a real number $\alpha \in [0,1)$ and $\beta \in (2,\infty)$ with $t := (\lfloor\beta\rfloor - \alpha - \beta + 1)/\beta \in [0, \lceil\beta\rceil/\beta - 1]$, define a map $T = T_{\alpha,\beta}\colon [0,1] \to [0,1]$ by

$$T_{\alpha,\beta}(x) := \begin{cases} \beta x - \lfloor \beta x \rfloor & (x \in [0, (\alpha+\beta-1)/\beta)), \\ \beta(x-1)+1 & (x \in [(\alpha+\beta-1)/\beta, 1]). \end{cases} \tag{5.1}$$

Then it is clear that $T_{\alpha,\beta}$ belongs to neither (II) nor (III). We remark that $T_{\alpha,\beta}$ coincides with the map S_t introduced by Akiyama, Kaneko and Kim in [2, Example 4.4]. Let k be the smallest integer, which is not less than $\alpha+\beta$, and set $a_i := i/\beta$ for $0 \le i \le k-2$, $a_{k-1} := (\alpha+\beta-1)/\beta$ and $a_k := 1$. Then it is straightforward to see that $T|_{(a_{i-1},a_i)}$ is strictly monotonic and continuous for each $1 \le i \le k$ and it follows from [5, Theorem 9.4.1] and [8, Proposition 3.6] that $h_{\mathrm{top}}(T) = \log\beta > 0$. Moreover, it is proved in [2, Appendix A] that T admits a unique measure of maximal entropy with full support, which implies that T is transitive. Let $\mathcal{I}\colon X_T \to \Sigma_k^+$ be the coding map and (Σ_T^+, σ) be the coding space of $([0,1],T)$. We set $\xi = \{\xi_n\}_{n\ge1}$ and $\zeta = \{\zeta_n\}_{n\ge1}$ in Σ_T^+ by

$$\xi := \lim_{x\to a_{k-1}+0, x\in X_T} \mathcal{I}(x) \text{ and } \zeta := \lim_{x\to a_{k-1}-0, x\in X_T} \mathcal{I}(x).$$

In what follows we will study the property of Hofbauer's Markov Diagram $(\mathcal{D}_T, \to)$ of T. First, we define sequences of closed sets $\{A_n\}_{n\ge1}$ and $\{B_n\}_{n\ge1}$ of Σ_T^+ inductively as follows: Let $A_1 := [\xi_1]$ and $B_1 := [\zeta_1]$. If A_n and B_n are defined for $n \ge 1$, then we set $A_{n+1} := \sigma(A_n) \cap [\xi_{n+1}]$ and $B_{n+1} := \sigma(B_n) \cap [\zeta_n]$. We will also define sequences of integers $\{r_n\}_{n\ge1}$ and $\{s_n\}_{n\ge1}$ inductively. Let $r_1 := \min\{j \in \mathbb{N} : \xi_{j+1} \ne k\} (= 1)$. If $r_1, \dots, r_n$ is defined for $n \ge 1$, then we set

$$r_{n+1} := \begin{cases} \min\{j \in \mathbb{N} : \xi_{r_1+\cdots+r_n+j} \ne k\} & (\xi_{r_1+\cdots+r_n+1} \ne k-1), \\ \max\{j \in \mathbb{N} : \xi_{r_1+\cdots+r_n+1}\cdots\xi_{r_1+\cdots+r_n+j} = \zeta_1\cdots\zeta_j\} & (\mathit{else}). \end{cases}$$

Similarly, let $s_1 := \min\{j \in \mathbb{N} : \zeta_{j+1} \neq 1\}$. If $s_1, \ldots, s_n$ is defined for $n \geq 1$, then we set

$$s_{n+1} := \begin{cases} \min\{j \in \mathbb{N} : \zeta_{s_1+\cdots+s_n+j} \neq 1\} & (\zeta_{s_1+\cdots+s_n+1} \neq k), \\ \max\{j \in \mathbb{N} : \zeta_{s_1+\cdots+s_n+1} \cdots \zeta_{s_1+\cdots+s_n+j} = \xi_1 \cdots \xi_j\} & (else). \end{cases}$$

For each $n \geq 1$, we set $R_n := r_1 + \cdots + r_n$ and $S_n := s_1 + \cdots + s_n$. Note that $\sigma[i] \cap [j] = [j]$ for any $1 \leq i \leq k-2$ and $1 \leq j \leq k$. This together with [11, (5.8)] imply the following proposition:

Proposition 5.1. ([11]) For any $n \geq 0$, we have $\mathcal{D}_n = \{[1], \ldots, [k-2]\} \cup \{A_j, B_j : j \leq n+1\}$, where $\mathcal{D}_n$ is as in §2. In particular, we have $\mathcal{D}_T = \{[1], \ldots, [k-2]\} \cup \{A_n, B_n : n \geq 1\}$. Moreover, the Markov Diagram has the following arrows:

- $[i] \to [j]$ for $1 \leq i \leq k-2$ and $1 \leq j \leq k$.
- $A_n \to A_{n+1}$ and $B_n \to B_{n+1}$ for any $n \geq 1$.
- For any $n \geq 1$,

$$A_{R_n} \to \begin{cases} [i] & \text{if } \xi_{R_{n-1}+1} \neq k-1 \text{ and } \xi_{R_n+1} < i \leq k, \\ B_{r_n+1} & \text{if } \xi_{R_{n-1}+1} = k-1, \\ [j] & \text{if } \xi_{R_{n-1}+1} = k-1 \text{ and } \xi_{R_n+1} < j < \zeta_{r_n+1}. \end{cases}$$

- For any $n \geq 1$,

$$B_{S_n} \to \begin{cases} [i] & \text{if } \zeta_{S_{n-1}+1} \neq k \text{ and } 1 \leq i < \zeta_{S_n+1}, \\ A_{s_n+1} & \text{if } \zeta_{S_{n-1}+1} = k, \\ [j] & \text{if } \zeta_{S_{n-1}+1} = k \text{ and } \xi_{s_n+1} < j < \zeta_{S_n+1}. \end{cases}$$

Here we set $R_0 := 0$ and $S_0 := 0$ for the convenience.

We define two subsets of integers $\kappa(\xi)$ and $\kappa(\zeta)$ by

$$\kappa(\xi) := \{n \in \mathbb{N} : \zeta_1 \cdots \zeta_n = \xi_j \cdots \xi_{j+n-1} \text{ for some } j \in \mathbb{N}\} \text{ and}$$

$$\kappa(\zeta) := \{n \in \mathbb{N} : \zeta_j \cdots \zeta_{j+n-1} = \xi_1 \cdots \xi_n \text{ or } 1^n \text{ for some } j \in \mathbb{N}\}.$$

The following proposition is essentially proved in [6] and so we omit the proof:

Proposition 5.2. ([6, 2.1 Proposition]) Suppose that $\kappa(\xi)$ is unbounded. Then T does not have the specification property.

Next, we give a sufficient condition to satisfy conditions (a)-(e).

Proposition 5.3. Suppose that $\kappa(\zeta)$ is bounded. Then T satisfies conditions (a)-(e).

Proof. Note that T is piecewise increasing and right continuous. Hence to prove the proposition, it is sufficient to prove the condition (d) by Theorem B. Since T is transitive and $h_{\mathrm{top}}(T) > 0$, it follows from [13, Theorem 11] that there is an irreducible subset $\mathcal{C} \subset \mathcal{D}_T$ such that $\Sigma_T^+ = \Psi(\Sigma_{\mathcal{C}}^+)$ holds. Moreover, since $\mathcal{D}_T = \{[1], \ldots, [k-2]\} \cup \{A_n, B_n : n \geq 1\}$, one can choose $\mathcal{C}$ so that $C \in \mathcal{C}$ and $C \to D$ imply $D \in \mathcal{C}$. Hence it follows from [13, Theorem 10] that there exsits $N_1 \in \mathbb{N}$ such that for any $x \in \Sigma_T^+$, we can find $(C_i)_{i\in\mathbb{N}} \in \Sigma_{\mathcal{C}}^+$ such that $\Psi((C_i)_{i\in\mathbb{N}}) = x$ and $C_1 \in \mathcal{D}_{N_1}$. Since $\kappa(\zeta)$ is bounded, it follows from Propostion 5.1 that there exists $N_2 \in \mathbb{N}$ such that for any $n \geq 1$, there is a finite path $C_1 \cdots C_j$ in $\mathcal{C}$ such that $j \leq N_2$, $C_1 = B_n$, and $C_j \in \mathcal{D}_{N_2}$. Let $N := \max\{N_1, N_2\}$. For $C, D \in \mathcal{C} \cap \mathcal{D}_N$, we set $t(C, D) := \min\{n \geq 1 : C_1 \cdots C_n$ is a finite path in $\mathcal{C}$ with $C_1 = C$ and $C_n = D\}$ and $M := \max\{t(C, D) : C, D \in \mathcal{C} \cap \mathcal{D}_N\} < \infty$.

Let $\mu^+ \in \mathcal{M}_\sigma^e(\Sigma_T^+)$. Since μ^+ is ergodic, we can find a point $x \in \Sigma_T^+$ such that $(\sum_{j=0}^{n} \delta_{\sigma^j(x)})/n$ converges to μ^+. Take an infinite path $(C_i)_{i\in\mathbb{N}} \in \Sigma_{\mathcal{C}}^+$ such that $\Psi((C_i)_{i\in\mathbb{N}}) = x$ and $C_1 \in \mathcal{D}_{N_1}$. Let $n \geq 1$. In what follows we will define a periodic infinite path $(E_i^n)_{i\in\mathbb{N}}$ in $\mathcal{C}$. We devide into two cases.

(*Case 1*) $C_n = A_j$ for some $j \geq 1$.
Let $l \geq 0$ be a minimal integer so that A_{j+l} has at least two successors. Then it is clear that $C_n + i = A_{j+i}$ for any $0 \leq i \leq l$. Choose one of the successor C of A_{j+l} so that $C \neq A_{j+l+1}$. Then it follows from Proposition 5.1 that $C \notin \{A_m : m \geq 1\}$. This together with $C_1 \in \mathcal{D}_N$ imply that there is a finite path $D_1 \cdots D_m$ in $\mathcal{C}$ such that $m \leq N + M$, $D_1 = D$ and $D_m = C_1$. Then we define $(E_i^n)_{i\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ by

$$E_i^n = \begin{cases} C_i & (1 \leq i \leq n + l), \\ D_{i-n-l} & (n + l + 1 \leq i \leq n + l + m - 1), \end{cases}$$

and $E_{i+q(n+l+m-1)}^n = E_i^n$ for any $q \geq 1$ and $1 \leq i \leq n + l + m - 1$.

(*Case 2*) $C_n \notin \{A_j : j \geq 1\}$.
In this case, it is easy to see that there is a finite path $D_1 \cdots D_m$ in $\mathcal{C}$ such that $j \leq N + M$, $D_1 = C_n$ and $D_m = C_1$. Then we define $(E_i^n)_{i\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ by

$$E_i^n = \begin{cases} C_i & (1 \leq i \leq n), \\ D_{i-n+1} & (n + 1 \leq i \leq n + m - 2), \end{cases}$$

and $E_{i+q(n+m-2)}^n = E_i^n$ for any $q \geq 1$ and $1 \leq i \leq n + m - 2$.

We set $z^n := \Psi((D_i^n)_{i\in\mathbb{N}})$. By the definition of z^n, it is easy to see that z^n is periodic and $P_\sigma(z^n)$ converges to μ^+, which imply the condition (d). □

Now, we give a piecewise monotonic map which does not belong to any of (I)-(III), but satisfies conditions (a)-(e).

Example 5.4. First, we consider the following equation:

$$\frac{-X^2 + 4X - 2}{X - 1} = \sum_{i=1}^{\infty} \frac{b_i - 1}{X^i}, \tag{5.2}$$

where $b = (b_i)_{i\in\mathbb{N}}$ is defined by $b = 41321322132221322221 \cdots = 41 \prod_{j=1}^{\infty} 32^j 1$. Then by the intermediate value theorem, we can see that the equation (5.2) has a unique positve real solution and denote the solution by $\beta = 3.2205 \cdots$. Denote $\alpha := (-\beta^2 + 4\beta - 2)/(\beta - 1)$ and let $T = T_{\alpha,\beta}$ be as in (5.1). Then it is easy to see that $\zeta = 3222 \cdots$ and by the equation (5.2), we have $\xi = b$.

Hence $\kappa(\xi)$ is unbounded and $\kappa(\zeta)$ is bounded. Therefore, by Propositions 5.2 and 5.3, T does not have the specification property but satisfies conditions (a)-(e).

6. Appendix

In this appendix, we prove that the transitivity of a piecewise monotonic map implies the injectivity of the coding map $\mathcal{I}\colon X_T \to \Sigma_k^+$ (Proposition 6.1). This is probably well-known and obtained by a slightly modification of the proof of [21, Theorem 3], but for the sake of completeness, we give a proof of the proposition.

Proposition 6.1. Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map. Then $\mathcal{I}\colon X_T \to \Sigma_T^+$ is injective.

Proof. By contradiction, assume that there exist two distinct points $y, y' \in X_T$ so that $(\omega_i)_{i\in\mathbb{N}} := \mathcal{I}(y) = \mathcal{I}(y')$. We set $x^- := \inf\{x \in X_T : \mathcal{I}(x) = (\omega_i)_{i\in\mathbb{N}}\}$ and $x^+ := \sup\{x \in X_T : \mathcal{I}(x) = (\omega_i)_{i\in\mathbb{N}}\}$, and let $I := (x^-, x^+)$. It is clear that for any $n \geq 1$, $T^n|_I$ is strictly monotonic and continuous, and hence $T^n(I)$ is an open interval. By the transitivity of T, we can find a minimal positive integer m such that $T^m(I) \cap I \neq \emptyset$, which implies that $\omega_i = \omega_{i+m}$ for any $i \geq 1$. Hence by the definition of I, we can see that $T^m(I) \subset I$. To prove the proposition, it is sufficient to show the existence of two open sets $U, V \subset [0,1]$ so that for any $n \geq 1$, $T^n(U) \cap V = \emptyset$ since this contradicts with the transitivity of T. If $T^m(I) \neq I$, then we can easily find an open interval $J \subset I$ so that $T^n(I) \cap J = \emptyset$ for any $n \geq 1$. Hence we may assume that $T^m(I) = I$.

(*Case 1*) $T^m|_I$ is increasing.

Take $x \in (x^-, x^+)$ and two open intervals $U_1 \subset (x^-, x)$ and $U_2 \subset (x, x^+)$.

(*Subcase 1.1*) $T^m(x) \leq x$.

In this case, it is easy to see that $T^m(U_1) \subset U_1$. Since m is a minimal positive integer so that $T^m(I) \cap I \neq \emptyset$, we have $T^i(U_1) \cap U_2 = \emptyset$ for $1 \leq i \leq m-1$. Therefore, we have $T^n(U_1) \cap U_2 = \emptyset$ for any $n \geq 1$.

(*Subcase 1.2*) $T^m(x) > x$.

We can prove $T^n(U_2) \cap U_1 = \emptyset$ for any $n \geq 1$ in a similar way to the proof of (*Subcase 1.1*).

(*Case 2*) $T^m|_I$ is decreasing.

Since $T^m(I) = I$, we have $T^m(x^+) = x^-$ and $T^m(x^-) = x^+$. Hence by the intermediate value theorem, we can find a point $x^- < x' < x^+$ such that $T^m(x') = x'$. Take $z \in (x^-, x')$ and two open intervals $U_1 \subset (x^-, z)$ and $U_2 \subset (z, x')$.

(*Subcase 2.1*) $T^{2m}(z) \leq z$.

It is clear that $T^m(U_1) \subset (x', x^+)$ and $T^{2m}(U_1) \subset U_1$. Hence for any $n \geq 1$, we have $T^n(U_1) \cap U_2 = \emptyset$.

(*Subcase 2.2*) $T^{2m}(z) > z$.

In this case, for any $n \geq 1$, we can show $T^n(U_2) \cap U_1 = \emptyset$ in a similar way to the proof of (*Subcase 2.1*).

Therefore, we have the proposition. □

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