DENSITY OF PERIODIC MEASURES AND LARGE DEVIATION PRINCIPLE FOR GENERALIZED MOD ONE TRANSFORMATIONS

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Abstract. We introduce a new class of piecewise monotonic maps, called generalized mod one transformations, which include all (α, β) and generalized β-transformations, and prove that all transitive generalized mod one transformations with positive topological entropy satisfy the level-2 large deviation principle with a unique measure of maximal entropy. This is obtained by our result on the density of periodic measures for more general class of piecewise mototonic maps, which is a partial answer to the open problem posed by Hofbauer-Raith [16, 23].

1. Introduction

In this paper, we consider piecewise monotonic maps on the unit interval $[0,1]$. We say that $T\colon [0,1] \to [0,1]$ is a *piecewise monotonic map* if there exist integer $k > 1$ and $0 = c_0 < c_1 < \cdots < c_k = 1$, which we call the *critical points*, such that $T|_{(c_{i-1},c_i)}$ is strictly monotonic and continuous for each $1 \leq i \leq k$. In most of this paper, we further assume the following conditions for a piecewise monotonic map T.

- The *topological entropy* $h_{\mathrm{top}}(T)$ of T is positive (see [1, Ch. 9] for the definition of topological entropy for piecewise monotonic maps).
- T is *topologically transitive*, that is, there exists a point $x \in [0,1]$ whose forward orbit $\{T^n(x) : n \geq 0\}$ is dense in $[0,1]$.

Under these conditions, it is proved in [12, Theorem 4] that there exists a unique T-invariant measure m of maximal entropy for T; that is, the metric entropy of m coincides with $h_{\mathrm{top}}(T)$. In this study, we investigate whether the large deviation principle holds for a piecewise monotonic map with the unique measure of maximal entropy as a reference.

Let $\mathcal{M}([0,1])$ be the set of all Borel probability measures on $[0,1]$ endowed with the weak*-topology. We say that $([0,1],T)$ satisfies the *(level-2) large deviation principle* with the unique measure of maximal entropy m as a reference if there exists a lower semi-continuous function $\mathcal{J}\colon \mathcal{M}([0,1]) \to [0,\infty]$, called a *rate function*, such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\left\{x \in [0,1] : \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)} \in \mathcal{K}\right\}\right) \leq -\inf_{\mathcal{K}} \mathcal{J}$$

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holds for any closed set $\mathcal{K} \subset \mathcal{M}([0,1])$ and

$$\liminf_{n\to\infty} \frac{1}{n} \log m \left(\left\{ x \in [0,1] : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} \in \mathcal{U} \right\} \right) \geq -\inf_{\mathcal{U}} \mathcal{J}$$

holds for any open set $\mathcal{U} \subset \mathcal{M}([0,1])$. Here δ_y signifies the Dirac mass at a point $y \in [0,1]$. We refer to [7, 8] for a general theory of large deviations and its background in statistical mechanics.

In such a situation, it was shown in [30, 34] that the large deviation principle holds if the map has the specification property (see [2, §1]for the definition of the specification property). Hence we focus our attention to piecewise monotonic maps without the specification property. A familiar example included in this family is a *β-transformation*.

- **β-transformations.**
  The β-transformation $T_\beta \colon [0,1] \to [0,1]$ with $\beta > 1$ was introduced by Rényi [25] and defined by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1 & (x \neq 1), \\ \lim\limits_{y \to 1-0} (\beta y \pmod 1) & (x = 1). \end{cases}$$

This family has recently attracted attention in the setting beyond specification, since T_β does not satisfy the specification property for Lebesgue almost parameter $\beta > 1$ ([2, 26]). After that, various generalizations of T_β has been considered by many authors: *(α, β)-transformations* ([4, 5, 9, 11, 24]), *$(-\beta)$-transformations* ([18, 19, 20, 27]), *generalized β-transformations* ([9, 10, 29, 32]) (See also Figure 1).

- **(α, β)-transformations.**
  The (α, β)-transformation $T_{\alpha,\beta} \colon [0,1] \to [0,1]$ with $\beta > 1$ and $0 \leq \alpha < 1$ was introduced by Parry ([21]) and defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \beta x + \alpha \pmod 1 & (x \neq 1), \\ \lim\limits_{y \to 1-0} (\beta y + \alpha \pmod 1) & (x = 1). \end{cases}$$

- **Generalized β-transformations.**
  Let $\beta > 1$ and k be a smallest integer, which is not less than β, and fix $E = (E_1, \ldots, E_k) \in \{+1, -1\}^k$. We partition $[0,1]$ into k intervals

$$I_1 := [0, 1/\beta),\ I_2 := [1/\beta, 2/\beta), \ldots, I_k := [k - 1/\beta, 1].$$

  The generalized β-transformation $T_{\beta,E} \colon [0,1] \to [0,1]$ was introduced by Góra ([10]) and defined by

$$T_{\beta,E}(x) = \begin{cases} \beta x - i + 1 & (x \in I_i,\ E(i) = +1), \\ -\beta x + i & (x \in I_i,\ E(i) = -1). \end{cases}$$

  If $E = (-1, \ldots, -1)$, then we call $T_{\beta,E}$ a $(-\beta)$-transformation.

Pfister and Sullivan ([22]) established the large deviation principle for the β-transformations for any $\beta > 1$, which is the first work on the large deviation principle in this family without the specification property. In [5],

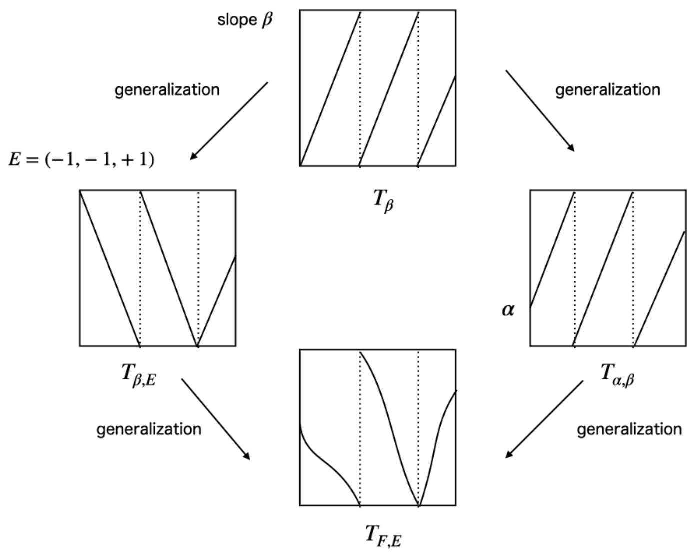

FIGURE 1. Graphs of T_β, $T_{\alpha,\beta}$, $T_{\beta,E}$ and $T_{F,E}$.

Chung and the second author proved that large deviation principle holds for (α, β) and generalized β-transformations in the following parameters:

- (α, β)-transformations for $0 \le \alpha < 1$ and $\beta > 2$;
- Generalized β-transformations for $\frac{1+\sqrt{5}}{2} < \beta < 2$ and $E = (-1, -1)$.

Then it is natural to ask whether does the large deviation principle hold for the other parameters. In this paper, not only we get an affirmative answer to the all parameters, but also we show that it holds for more general class of piecewise monotonic maps, which we call *generalized mod one transformations*.

- **Generalized mod one transformations**
  Let $F : [0, 1] \to \mathbb{R}$ be a strictly increasing and continuous map such that $F(0) \in [0, 1)$. Let k be a smallest integer which is not less than $F(1)$. Fix $(E_1, \ldots, E_k) \in \{+1, -1\}^k$ and consider k intervals
  $$I_1 := [0, c_1), I_2 := [c_1, c_2), \ldots, I_k := [c_{k-1}, 1]$$
  where $\{c_i\} = F^{-1}\{i\}$ for $1 \le i \le k-1$. We define the generalized mod one transformation $T_{F,E} : [0, 1] \to [0, 1]$ by
  $$T_{F,E}(x) = \begin{cases} F(x) - i + 1 & (x \in I_i, E_i = +1) \\ -F(x) + i & (x \in I_i, E_i = -1). \end{cases}$$

Generalized mod one transformations are clearly a generalization of both (α, β)-transformations and generalized β-transformations. The graphs of T_β, $T_{\alpha,\beta}$, $T_{\beta,E}$, $T_{F,E}$ are plotted in Figure 1. Now, we state our main result of this paper.

Theorem A. *Let $T\colon [0,1] \to [0,1]$ be a transitive generalized mod one transformation with $h_{\mathrm{top}}(T) > 0$. Then $([0,1], T)$ satisfies the level-2 large deviation principle with the unique measure of maximal entropy as a reference, and the rate function $\mathcal{J}\colon \mathcal{M}([0,1]) \to [0,\infty]$ satisfies*

$$\mathcal{J}(\mu) = \begin{cases} h_{\mathrm{top}}(T) - h_T(\mu) & (\mu \text{ is } T\text{-invariant and } \mu(C_T) = 0), \\ \infty & (\mu \text{ is not } T\text{-invariant and } \mu(C_T) = 0). \end{cases} \tag{1.1}$$

Here, $C_T := \{c_0, \ldots, c_k\}$ is the set of critical points and $h_T(\mu)$ denotes the metric entropy of μ.

Remark 1.1. In fact, the rate function $\mathcal{J}$ in Theorem A coincides with that in [5, Theorem A] given by [5, (3.5)] from which the equation (1.1) follows immediately. We also remark that $\mathcal{J}(\mu)$ could be either finite or infinite value if $\mu(C_T) > 0$.

A crucial step to prove Theorem A is to show the density of periodic measures. For a metrizable space X and Borel measurable map $f\colon X \to X$, denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X endowed with the weak*-topology, by $\mathcal{M}_f(X) \subset \mathcal{M}(X)$ the set of f-invariant ones, and by $\mathcal{M}_f^e(X) \subset \mathcal{M}_f(X)$ the set of ergodic ones. We say that $\mu \in \mathcal{M}(X)$ is a *periodic measure* if there exist $x \in X$ and $n > 0$ such that $f^n(x) = x$ and $\mu = \delta_n^f(x) := 1/n \sum_{j=0}^{n-1} \delta_{f^j(x)}$ hold. Then, it is clear that $\mu \in \mathcal{M}_f^e(X)$. We denote by $\mathcal{M}_f^p(X) \subset \mathcal{M}_f^e(X)$ the set of all periodic measures on X. It is established by Chung and the second author that the level-2 large deviation principle for a piecewise monotonic map is followed by the density of periodic measures with the irreducibility of its Markov Diagram (see [5, Theorem A]). We note that the irreducibility is slightly stronger than the transitivity. In this paper, we show that this result remains true if the irreducibility of a Markov Diagram is replaced by the transitivity of a map.

Proposition A. *Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map with $h_{\mathrm{top}}(T) > 0$ and suppose that $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T^e([0,1])$. Then $([0,1], T)$ satisfies the level-2 large deviation principle with the unique measure of maximal entropy as a reference, and the rate function $\mathcal{J}\colon \mathcal{M}([0,1]) \to [0,\infty]$ is expressed by [5, (3.5)]. In particular, $\mathcal{J}$ satisfies the equation (1.1) (see Remark 1.1).*

Hence Theorem A follows from Proposition A and the following theorem, which is the second result of this paper.

Theorem B. *Let $T\colon [0,1] \to [0,1]$ be a transitive generalized mod one transformation with $h_{\mathrm{top}}(T) > 0$. Then $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T^e([0,1])$.*

Our proof of theorem B is based on the work by Hofbauer and Raith ([16]) where the density of periodic measures proved for a piecewise monotonic map consisting of two monotonic pieces. Since a generalized mod one transformation can have more than three monotonic pieces, more complicated method is needed. One key difference between [16] and ours is Proposition 3.4, which is one of the novelty of this paper (see also Remark 3.5).

Remark 1.2. We emphasize that Theorem B can be seen as a special case of our more general result (Theorem C in §4), which provides sufficient conditions for piecewise monotonic maps to satisfy density of periodic measures in ergodic ones (see for the details in §4).

The remainder of this paper is organized as follows. In §2, we establish our definitions and prepare several facts. Subsequently, we present proofs of Proposition A and Theorem B in §3. In §4 we illustrate Theorem C and further problems.

2. Preliminaries

2.1. **Symbolic dynamics.** Let $\mathbb{N}_0$ be the set of non-negative integers. For a finite or countable set A, we denote by $A^{\mathbb{N}_0}$ the one-sided infinite product of A equipped with the product topology of the discrete topology of A. To simplify the notation, given integers $i \le j$ and $x_i, x_{i+1}, \dots, x_j \in A$, we set $x_{[i,j]} := x_i x_{i+1} \cdots x_j$. We also write $x_{[i,j)} := x_{[i,j-1]}$ and similarly for $x_{(i,j]}$ and $x_{(i,j)}$. A sequence $\underline{x} \in A^{\mathbb{N}_0}$ will be defined by all of its coordinates $x_n \in A$ with $n \ge 0$. Let σ be the shift map on $A^{\mathbb{N}_0}$ (i.e., $(\sigma(\underline{x}))_n = x_{n+1}$ for each $n \ge 0$ and $\underline{x} \in A^{\mathbb{N}_0}$). When a subset Σ^+ of $A^{\mathbb{N}_0}$ is σ-invariant and closed, we call it a *subshift* and call A the *alphabet* of Σ^+. For a matrix $M = (M_{ij})_{(i,j)\in A^2}$, each entry of which is 0 or 1, we define a subshift $\Sigma_M^+ \subset A^{\mathbb{N}_0}$ by

$$\Sigma_M^+ = \{\underline{x} \in A^{\mathbb{N}_0} : M_{x_n x_{n+1}} = 1 \text{ for all } n \ge 0\}$$

and call Σ_M^+ a *Markov shift* with an *adjacency matrix* M.

For a subshift Σ^+ on an alphabet A, we set $\mathcal{L}(\Sigma^+) := \{x_{[0,n]} : \underline{x} \in \Sigma^+,\ n \ge 0\}$ and $[u] := \left\{\underline{x} \in \Sigma^+ : u = x_{[0,|u|)}\right\}$ for each $u \in \mathcal{L}(\Sigma^+)$, where $|u|$ denotes the length of u. A word $v \in \mathcal{L}(\Sigma^+)$ is called a *subword* of $u = u_0 \cdots u_n \in \mathcal{L}(\Sigma^+)$ if $v = u_{[i,j]}$ for some $0 \le i \le j \le n$. For $u, v \in \mathcal{L}(\Sigma^+)$, we use juxtaposition uv to denote the word obtained by the concatenation and u^∞ means a one-sided infinite sequence $uuu \cdots \in A^{\mathbb{N}_0}$. Moreover, we set $\mathcal{S}(u) := \{i \in A : ui \in \mathcal{L}(\Sigma^+)\}$ for $u \in \mathcal{L}(\Sigma^+)$. Finally, we say that Σ^+ is *transitive* if for any $u, v \in \mathcal{L}(\Sigma^+)$, we can find $w \in \mathcal{L}(\Sigma^+)$ such that $uwv \in \mathcal{L}(\Sigma^+)$ holds. For the rest of this paper, we denote by $h_\sigma(\mu)$ the *metric entropy* of $\mu \in \mathcal{M}_\sigma(\Sigma^+)$.

2.2. **Markov diagram.** Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with critical points $0 = c_0 < c_1 < \cdots < c_k = 1$. Let $X_T := \bigcap_{n=0}^\infty T^{-n}(\bigcup_{j=1}^k (c_{j-1}, c_j))$, and define the *coding map* $\mathcal{I}\colon X_T \to \{1, \dots, k\}^{\mathbb{N}_0}$ by

$$(\mathcal{I}(x))_n = j \text{ if and only if } T^n(x) \in (c_{j-1}, c_j).$$

We note that the coding map $\mathcal{I}$ is injective if T is transitive (see [33, Proposition 6.1]). We denote the closure of $\mathcal{I}(X_T)$ in $\{1, \dots, k\}^{\mathbb{N}}$ by Σ_T^+. Then,

Σ_T^+ is a subshift, and (Σ_T^+, σ) is called the *coding space* of $([0,1], T)$. We use the following notations:

- $\underline{a}^{(i)} := \lim_{x \to c_i + 0} \mathcal{I}(x)$, $\underline{b}^{(i)} := \lim_{x \to c_i - 0} \mathcal{I}(x)$ for $1 \le i \le k-1$.
- $\mathrm{adj}(\underline{a}^{(i)}) := \underline{b}^{(i)}$, $\mathrm{adj}(\underline{b}^{(i)}) := \underline{a}^{(i)}$ for $1 \le i \le k-1$.
- $\underline{a} := \lim_{x \to +0} \mathcal{I}(x)$, $\underline{b} := \lim_{x \to 1-0} \mathcal{I}(x)$.

We also set $\mathcal{CR} := \{\underline{a}^{(i)}, \underline{b}^{(i)} : 1 \le i \le k-1\}$ and call it a *critical set*.

In what follows, we define the Markov diagram, introduced by Hofbauer ([12]), which is a countable oriented graph with subsets of Σ_T^+ as vertices. Let $C \subset \Sigma_T^+$ be a closed subset with $C \subset [j]$ for some $1 \le j \le k$. We say that a non-empty closed subset $D \subset \Sigma_T^+$ is a *successor* of C if $D = [l] \cap \sigma(C)$ for some $1 \le l \le k$. The expression $C \to D$ denotes that D is a successor of C. Now, we define a set $\mathcal{D}_T$ of vertices by induction. First, we set $\mathcal{D}_0 := \{[1], \dots, [k]\}$. If $\mathcal{D}_n$ is defined for $n \ge 0$, then we set

$$\mathcal{D}_{n+1} := \mathcal{D}_n \cup \{D : D \text{ is a successor for some } C \in \mathcal{D}_n\}.$$

We note that $\mathcal{D}_n$ is a finite set for each $n \ge 0$ since the number of successors of any closed subset of Σ_T^+ is at most k by definition. Finally, we set

$$\mathcal{D}_T := \bigcup_{n \ge 0} \mathcal{D}_n.$$

The oriented graph $(\mathcal{D}_T, \to)$ is called the *Markov diagram* of T. For notational simplicity, we use the expression $\mathcal{D}$ instead of $\mathcal{D}_T$ if no confusion arises.

For $\underline{x} \in \Sigma_T^+$ and $n \ge 0$, we set $D_n^{\underline{x}} := \sigma^n(x_{[0,n]})$. We define a sequence $\{R_m^{\underline{x}}\}_{m \ge 0}$ of integers inductively as follows. First, we set $R_0^{\underline{x}} := 0$. If $R_m^{\underline{x}}$ is defined for $m \ge 0$ and $R_m^{\underline{x}} < \infty$, then let

$$R_{m+1}^{\underline{x}} := \min\{n > R_m^{\underline{x}} : \#(\mathcal{S}(x_{[0,n)})) \ge 2\},$$

where we set $R_{m+1}^{\underline{x}} = \infty$ if $\#(\mathcal{S}(x_{[0,n)})) = 1$ for any $n > R_m^{\underline{x}}$. We also set $r_m^{\underline{x}} := R_m^{\underline{x}} - R_{m-1}^{\underline{x}}$ if $R_m^{\underline{x}} < \infty$.

Remark 2.1. We remark that if the coding map $\mathcal{I} \colon X_T \to \Sigma_T^+$ is injective, then the cardinality of the cylinder set $[u]$ is $2^{\aleph_0}$ for any $u \in \mathcal{L}(\Sigma_T^+)$, which implies that $R_m^{\underline{x}} < \infty$ for any $m \ge 0$ and $\underline{x} \in \Sigma_T^+$. In particular, this finiteness holds if T is transitive.

Now, we summarize properties of Hofbauer's Markov Diagram, which are appeared in [14] (see also [16, Page 224]).

Proposition 2.2. Let $T \colon [0,1] \to [0,1]$ be a piecewise monotonic map and $(\mathcal{D}, \to)$ be the Markov diagram of T.

(1) $\mathcal{D} = \{D_n^{\underline{x}} : \underline{x} \in \mathcal{CR} \cup \{\underline{a}, \underline{b}\},\ n \ge 0\}$.

(2) $(\mathcal{D}, \to)$ has the following arrows:

- For any $\underline{x} \in \mathcal{CR} \cup \{\underline{a}, \underline{b}\}$ and any $n \ge 0$, $D_n^{\underline{x}} \to D_{n+1}^{\underline{x}}$.

- For any $\underline{x} \in \mathcal{CR} \cup \{\underline{a}, \underline{b}\}$ and any $m \geq 1$ with $R_m^{\underline{x}} < \infty$, $D^{\underline{x}}_{R_m^{\underline{x}}-1} \to D^{f_m(\underline{x})}_{r_m^{\underline{x}}}$. Here $f_m(\underline{x})$ is a unique point in $\mathcal{CR}$ so that $f_m(\underline{x}) \neq \sigma^{R_{m-1}^{\underline{x}}}(\underline{x})$ and $f_m(\underline{x}) \in D^{\underline{x}}_{R_{m-1}}$ hold.

(3) Let $\underline{x} \in \mathcal{CR} \cup \{\underline{a}, \underline{b}\}$ and $m \geq 1$, which satisfy $R_m^{\underline{x}} < \infty$.

- For any $0 \leq n \leq r_m^{\underline{x}} - 1$, we have $D^{\underline{x}}_{n+R_{m-1}} \subset D_n^{f_m(\underline{x})}$. In particular, $x_{[R_{m-1}^{\underline{x}}, R_m^{\underline{x}})} = f_m(\underline{x})_{[0, r_m^{\underline{x}})}$ holds.
- There exists an integer q such that $r_m^{\underline{x}} = R_q^{f_m(\underline{x})}$.

(4) If $C \in \mathcal{D}$ and $\#\mathcal{S}(C) > 2$, then $\mathcal{S}(C) \cap \mathcal{D}_0 \neq \emptyset$.

For a subset $\mathcal{C} \subset \mathcal{D}$, we define a matrix $M(\mathcal{C}) = (M(\mathcal{C})_{C,D})_{(C,D) \in \mathcal{C}^2}$ by

$$M(\mathcal{C})_{C,D} = \begin{cases} 1 & (C \to D), \\ 0 & (\text{otherwise}). \end{cases}$$

Then, $\Sigma^+_{M(\mathcal{C})} = \{\underline{C} \in \mathcal{C}^{\mathbb{N}_0} : C_n \to C_{n+1}, n \geq 0\}$ is a one-sided Markov shift with a countable alphabet $\mathcal{C}$ and an adjacency matrix $M(\mathcal{C})$. For notational simplicity, we denote $\Sigma^+_{\mathcal{C}}$ instead of $\Sigma^+_{M(\mathcal{C})}$.We say that $\mathcal{C}$ is *irreducible* if for any $C, D \in \mathcal{C}$, there are finite vertices $C_0, \ldots, C_n \in \mathcal{C}$ such that $C_i \to C_{i+1}$ for $0 \leq i \leq n-1$ (that is, $C_{[0,n]} \in \mathcal{L}(\Sigma^+_{\mathcal{C}})$), $C_0 = C$ and $C_n = D$, and if every subset of $\mathcal{D}$, which contains $\mathcal{C}$ does not have this property. It is clear that $\Sigma^+_{\mathcal{C}}$ is transitive if and only if $\mathcal{C}$ is irreducible. We define a map $\Psi \colon \Sigma^+_{\mathcal{D}} \to \{1, \ldots, k\}^{\mathbb{N}_0}$ by

$$\Psi((C_n)_{n \in \mathbb{N}_0}) := (x_n)_{n \in \mathbb{N}_0} \text{ for } (C_n)_{n \in \mathbb{N}_0} \in \Sigma^+_{\mathcal{D}},$$

where $1 \leq x_n \leq k$ is a unique integer such that $C_n \subset [x_n]$ holds for each $n \in \mathbb{N}_0$. Then it is not difficult to see that Ψ is continuous, countable to one and $\Psi(\Sigma^+_{\mathcal{D}}) = \Sigma^+_T$. We say that $\mu \in \mathcal{M}^e_\sigma(\Sigma^+_T)$ is *liftable* if there is $\overline{\mu} \in \mathcal{M}^e_\sigma(\Sigma^+_{\mathcal{D}})$ such that $\mu = \overline{\mu} \circ \Psi^{-1}$ holds. It is known that not every $\mu \in \mathcal{M}^e_\sigma(\Sigma^+_T)$ is liftable in general although Ψ is surjective (see [17]). In [12], Hofbauer provided a sufficient condition for the liftability.

Lemma 2.3. ([12, Lemma 3]) If $\mu \in \mathcal{M}^e_\sigma(\Sigma^+_T)$ has positive metric entropy, then μ is liftable.

We recall two important facts for transitive piecewise monotonic maps with positive topological entropy.

Lemma 2.4. ([14, Theorem 11],[16, Page 224]) There exists an irreducible subset $\mathcal{C} \subset \mathcal{D}$ satisfying the following properties:

- $\Psi(\Sigma^+_{\mathcal{C}}) = \Sigma^+_T$.
- $C \in \mathcal{C}$ and $C \to D$ implies $D \in \mathcal{C}$.
- There exists an integer n_0 such that $D^{\underline{x}}_{n_0} \in \mathcal{C}$ holds for any $\underline{x} \in \mathcal{CR} \cup \{\underline{a}, \underline{b}\}$.

Theorem 2.5. ([16, Theorem 1]) Let $T \colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map with $h_{\mathrm{top}}(T) > 0$. Suppose that there are integers N_0

and N_1 such that for any $\underline{x} \in \mathcal{CR}$, and any $j \in \mathbb{N}$ with $R_j^{\underline{x}} > N_0$, there exist an integer $1 \le m < j$, a periodic point $\underline{p}$ with period l and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \ge R_j^{\underline{x}} - R_m^{\underline{x}} - N_1$ such that u is a subword of both $x_{[R_m^{\underline{x}}, R_j^{\underline{x}})}$ and $p_{[0,l)}$. Then $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T^e([0,1])$.

Hereafter, let $T\colon [0,1] \to [0,1]$ be a generalized mod one transformation. To simplifies the notation, we set $A_n := \sigma^n([a_{[0,n]}])$ and $B_n := \sigma^n([b_{[0,n]}])$ for each $n \ge 0$ and set $R_m := R_m^{\underline{a}}$ and $S_m := R_m^{\underline{b}}$ for each $m \ge 0$. We also set $r_{m+1} := r_{m+1}^{\underline{a}} = R_{m+1} - R_m$ and $s_{m+1} := r_{m+1}^{\underline{b}} = S_{m+1} - S_m$ if $R_{m+1}, S_{m+1} < \infty$. We denote

$$\mathcal{A}_1 := \{m \in \mathbb{N} : R_m < \infty,\ \sigma(f_m(\underline{a})) = \underline{a}\},$$
$$\mathcal{A}_2 := \{m \in \mathbb{N} : R_m < \infty,\ \sigma(f_m(\underline{a})) = \underline{b}\},$$
$$\mathcal{B}_1 := \{m \in \mathbb{N} : S_m < \infty,\ \sigma(f_m(\underline{b})) = \underline{a}\} \text{ and}$$
$$\mathcal{B}_2 := \{m \in \mathbb{N} : S_m < \infty,\ \sigma(f_m(\underline{b})) = \underline{b}\}.$$

Note that $\sigma[i] = \Sigma_T^+$ for each $2 \le i \le k-1$ and $\sigma(\underline{x}) \in \{\underline{a}, \underline{b}\}$ for any $\underline{x} \in \mathcal{CR}$. These together with Proposition 2.2 and Theorem 2.5 imply the following:

Proposition 2.6. Let $T\colon [0,1] \to [0,1]$ be a generalized mod one transformation and $(\mathcal{D}, \to)$ be the Markov diagram of T.

(1) $\mathcal{D} = \{A_n, B_n : n \ge 0\} \cup \{[2], \dots, [k-1]\}$.

(2) (i) For any $n \ge 0$, $A_n \to A_{n+1}$ and $B_n \to B_{n+1}$.

(ii) For $m \ge 1$,

- $A_{R_m - 1} \to A_{r_m - 1}$ and $a_{(R_{m-1}, R_m)} = a_{[0, r_m - 1)}$ if $m \in \mathcal{A}_1$,
- $A_{R_m - 1} \to B_{r_m - 1}$ and $a_{(R_{m-1}, R_m)} = b_{[0, r_m - 1)}$ if $m \in \mathcal{A}_2$.

(iii) For $m \ge 1$,

- $B_{S_m - 1} \to A_{s_m - 1}$ and $b_{(S_{m-1}, S_m)} = a_{[0, r_m - 1)}$ if $m \in \mathcal{B}_1$,
- $B_{S_m - 1} \to B_{s_m - 1}$ and $b_{(S_{m-1}, S_m)} = b_{[0, s_m - 1)}$ if $m \in \mathcal{B}_2$.

(3) There exist two maps $P\colon \mathcal{A}_2 \to \mathbb{N}_0$ and $Q\colon \mathcal{B}_1 \to \mathbb{N}_0$ such that for any $m \ge 1$, $r_m - 1 = S_{p(m)}$ and $s_m - 1 = R_{Q(m)}$.

Theorem 2.7. Let $T\colon [0,1] \to [0,1]$ be a transitive generalized mod one transformation with $h_{\mathrm{top}}(T) > 0$. Suppose that there are integers N_0 and N_1 satisfying the following conditions:

- For any $j \in \mathbb{N}$ with $R_j > N_0$, there is an integer $1 \le m < j$, a periodic point $\underline{p}$ with period l and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \ge R_j - R_m - N_1$ such that u is a subword of both $a_{[R_m, R_j)}$ and $p_{[0,l)}$.
- For any $j \in \mathbb{N}$ with $S_j > N_0$, there is an integer $1 \le m < j$, a periodic point $\underline{p}$ with period l and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \ge S_j - S_m - N_1$ such that u is a subword of both $b_{[R_m, R_j)}$ and $p_{[0,l)}$.

Then $\mathcal{M}_T^p([0,1])$ is dense in $\mathcal{M}_T^e([0,1])$.

3. Proofs

3.1. **Proof of Proposition A.** In this subsection, we give a proof of Proposition A with the assumption that T is transitive. As we mentioned in §1, this theorem appears as [5, Theorem A] with the stronger assumption that $\mathcal{D}$ is irreducible. In [5], the hypothesis of the irreducibility for $\mathcal{D}$ is used only in [5, Proposition 3.1]. The other part of the proof of [5, Theorem A] can be shown similarly by using the transitivity of T instead of the irreducibility. Hence to prove Proposition A, it is sufficient to show the following proposition, which is analogous to [5, Proposition 3.1].

Proposition 3.1. Let $T\colon [0,1] \to [0,1]$ be as in Proposition A and let $\mathcal{C} \subset \mathcal{D}$ be as in Lemma 2.4. Then for any $\epsilon > 0$, any $\mu \in \mathcal{M}_\sigma(\Sigma_T^+)$, and any neighborhood $\mathcal{U} \subset \mathcal{M}(\Sigma_T^+)$ of μ, there exist a finite set $\mathcal{F} \subset \mathcal{C}$ and $\rho \in \mathcal{M}_\sigma^e(\Psi(\Sigma_\mathcal{F}^+))$ such that $\rho \in \mathcal{U}$ and $h_\sigma(\rho) \geq h_\sigma(\mu) - 2\epsilon$.

Proof. Without loss of generality, we may assume that $h_\sigma(\mu) - \epsilon > 0$, otherwise the conclusion is yield by the assumption that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma^e(\Sigma_T^+)$. By Ergodic Decomposition Theorem and the affinity of the entropy map, there exists a finite convex combination of ergodic measures $\nu := \sum_{i=1}^p a_i\nu_i$ such that $h_\sigma(\nu) \in \mathcal{U}$ and $h_\sigma(\nu) \geq h_\sigma(\mu) - \epsilon$. Again by density of $\mathcal{M}_\sigma^p(\Sigma_T^+)$ in $\mathcal{M}_\sigma^e(\Sigma_T^+)$, we may assume $\nu_i \in \mathcal{M}_\sigma^p(\Sigma_T^+)$ whenever $h_\sigma(\nu_i) = 0$. Then we need the following lemma.

Lemma 3.2. For each $1 \leq i \leq p$, there exists $\overline{\nu_i} \in \mathcal{M}_\sigma^e(\Sigma_\mathcal{C}^+)$ such that $\nu_i = \overline{\nu_i} \circ \Psi^{-1}$ holds. In particular, $\overline{\nu} := \sum_{i=1}^p a_i\overline{\nu_i}$ satisfies $\overline{\nu} \in \mathcal{M}_\sigma(\Sigma_\mathcal{C}^+)$ and $\nu = \overline{\nu} \circ \Psi^{-1}$.

Proof. We divide the proof into two cases.

(Case 1) $h_\sigma(\nu_i) = 0$. In this case, ν_i is a periodic measure. Take a periodic point $\underline{x} \in \Sigma_T^+$ in the support of ν_i. Then it follows from [14, Theorem 8] and $\Psi(\Sigma_\mathcal{C}^+) = \Sigma_T^+$ that there are finite vertices $C_0, \dots, C_{n-1} \in \mathcal{C}$ such that $(C_{[0,n)})^\infty \in \Sigma_\mathcal{C}^+$ and $\Psi((C_{[0,n)})^\infty) = \underline{x}$. Hence if we set $\overline{\nu_i} := \delta_n^\sigma((C_{[0,n)})^\infty)$, then we have $\overline{\nu_i} \in \mathcal{M}_\sigma^e(\Sigma_\mathcal{C}^+)$ and $\nu_i = \overline{\nu_i} \circ \Psi^{-1}$.

(Case 2) $h_\sigma(\nu_i) > 0$. Since $\nu_i \in \mathcal{M}_\sigma^e(\Sigma_T^+)$ and $h_\sigma(\nu_i) > 0$, it follows from Lemma 2.3 that there exists $\overline{\nu_i} \in \mathcal{M}_\sigma^e(\Sigma_\mathcal{D}^+)$ such that $\overline{\nu_i} := \nu_i \circ \Psi^{-1}$ holds. Since $\overline{\nu_i}$ is ergodic, there exists an irreducible subset $\mathcal{C}' \subset \mathcal{D}$ such that $\overline{\nu}_i(\Sigma_{\mathcal{C}'}^+) = 1$. Assume by contradiction that $\mathcal{C}' \neq \mathcal{C}$. Since $\Sigma_{\mathcal{C}'}^+ \subset \Psi^{-1}(\Psi(\Sigma_{\mathcal{C}'}^+))$, we have $\nu_i(\Psi(\Sigma_{\mathcal{C}'}^+)) = \overline{\nu_i} \circ \Psi^{-1}(\Psi(\Sigma_{\mathcal{C}'}^+)) \geq \overline{\nu_i}(\Sigma_{\mathcal{C}'}^+) = 1$. This implies that $\nu_i(\Psi(\Sigma_\mathcal{C}^+) \cap \Psi(\Sigma_{\mathcal{C}'}^+)) = \nu_i(\Sigma_T^+ \cap \Psi(\Sigma_{\mathcal{C}'}^+)) = 1$. On the other hand, by [13, Theorem 1 (ii)], $\Psi(\Sigma_\mathcal{C}^+) \cap \Psi(\Sigma_{\mathcal{C}'}^+)$ is either empty or finite (see also [14, Page 385]). Hence by the ergodicity of ν_i, we have $h_\sigma(\nu_i) = 0$, which is a contradiction. □

Note that $\overline{\nu}$ is an invariant measure on a transitive countable Markov shift $\Sigma_\mathcal{C}^+$. Hence by the continuity of Ψ and [31, Main Theorem], we can find a finite set $\mathcal{F} \subset \mathcal{C}$ and an ergodic measure $\overline{\rho}$ on $\Sigma_{\mathcal{F}}^+$ such that $h_\sigma(\overline{\rho}) \geq h_\sigma(\overline{\nu}) - \epsilon$ and $\rho := \overline{\rho} \circ \Psi^{-1} \in \mathcal{U}$. Since $\Psi \colon \Sigma_{\mathcal{C}}^+ \to \Sigma_T^+$ is countable to one, by [3, Proposition 2.8], we have $h_\sigma(\nu) = h_\sigma(\overline{\nu})$ and $h_\sigma(\rho) = h_\sigma(\overline{\rho})$, which prove the proposition. □

Remark 3.3. Proposition 3.1 states that the density of periodic measures for the coding space of piecewise monotonic maps also implies the entropy-density of ergodic measures, that is, any invariant measure can be approximated by ergodic ones with close entropies to that of the given one. We should mention that the entropy-density implies various statistical properties as well as large deviations (see [6, Theorem 1]).

3.2. **Proof of Theorem B.** The aim of this subsection is to give a proof of Theorem B. First, we recall that $R_m < \infty$ and $S_m < \infty$ for all $m \geq 0$ by the transitivity of T (see Remark 2.1). Let n_0 be as in Lemma 2.4 and m_0 be a smallest integer such that $R_{m_0}, S_{m_0} \geq n_0$ holds. We set $N_0 := \max\{R_{m_0},\ S_{m_0}\}$ and let n_1 be a smallest number such that for any $C \in \mathcal{C} \cap \mathcal{D}_{n_0}$ and $D \in \mathcal{C} \cap \mathcal{D}_{N_0}$, there exist finite vertices $C_0, \dots, C_n$ with $n \leq n_1$ such that $C_{[0,n]} \in \mathcal{L}(\Sigma_{\mathcal{C}}^+)$, $C_0 = C$ and $C_n = D$. We note that $n_1 < \infty$ since $\mathcal{C}$ is irreducible. We set $N_1 := N_0 + n_1$. By Theorem 2.7, to prove Theorem B, it is sufficient to show the following:

(I) For any $j \in \mathbb{N}$ with $R_j > N_0$, there is an integer $1 \leq m < j$, a periodic point $\underline{p}$ with period l and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \geq R_j - R_m - N_1$ such that u is a subword of both $a_{[R_m, R_j)}$ and $p_{[0,l)}$.

(II) For any $j \in \mathbb{N}$ with $S_j > N_0$, there is an integer $1 \leq m < j$, a periodic point $\underline{p}$ with period l and $u \in \mathcal{L}(\Sigma_T^+)$ with $|u| \geq S_j - S_m - N_1$ such that u is a subword of both $b_{[R_m, R_j)}$ and $p_{[0,l)}$.

We only prove the item (I) because (II) can be shown in a similar manner. Take any $j \in \mathbb{N}$ with $R_j > N_0$. In what follows we will decompose the set $\mathbb{N}$ into six sets. Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be as in §2.3 and set

$$\mathcal{A}_3 := \{m \in \mathbb{N} : \mathcal{S}(A_{R_m - 1}) \cap \mathcal{D}_0 \neq \emptyset\} \text{ and}$$
$$\mathcal{B}_3 := \{m \in \mathbb{N} : \mathcal{S}(B_{S_m - 1}) \cap \mathcal{D}_0 \neq \emptyset\}.$$

First, it is clear that $\mathbb{N} = \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}_1 \cup (\mathcal{A}_2 \setminus \mathcal{A}_3) \cup \mathcal{A}_3$. We set

$$\mathcal{A}_2^{(1)} := \{m \in \mathcal{A}_2 : m+1 \in \mathcal{A}_1\}, \qquad \mathcal{A}_2^{(2)} := \{m \in \mathcal{A}_2 : P(m) \in \mathcal{B}_3\},$$
$$\mathcal{A}_2^{(3)} := \{m \in \mathcal{A}_2 : P(m)+1 \in \mathcal{B}_2\}, \text{ and } \mathcal{A}_4 := \mathcal{A}_2 \setminus (\mathcal{A}_3 \cup \textstyle\bigcup_{j=1}^3 \mathcal{A}_2^{(j)}),$$

noting that the map P is defined on $\mathcal{A}_2$ by Proposition 2.6 (3).
Then we have $\mathcal{A}_2 \setminus \mathcal{A}_3 \subset \bigcup_{j=1}^3 \mathcal{A}_2^{(j)} \cup \mathcal{A}_4$, which implies that

$$\mathbb{N} = \mathcal{A}_1 \cup \bigcup_{j=1}^{3} \mathcal{A}_2^{(j)} \cup \mathcal{A}_3 \cup \mathcal{A}_4.$$

Hence we can divide the proof into six cases:

(Case A) $j \in \mathcal{A}_1$**.** **(Case B)** $j \in \mathcal{A}_2^{(1)}$**.** **(Case C)** $j \in \mathcal{A}_2^{(2)}$**.**

(Case D) $j \in \mathcal{A}_2^{(3)}$**.** **(Case E)** $j \in \mathcal{A}_3$**.** **(Case F)** $j \in \mathcal{A}_4$.

We note that the proofs of **(Case A)** and **(Case B)** are similar to those of **(Case 1)** and **(Case 2)** in [16] respectively.

(Case C) $j \in \mathcal{A}_2^{(2)}$**.** Take $D \in \mathcal{S}(B_{S_{P(j)}-1}) \cap \mathcal{D}_0$. By the definition of n_1, we can find an integer $n \leq n_1$ and finite vertices $C_0, \ldots, C_n \in \mathcal{C}$ such that $C_{[0,n]} \in \mathcal{L}(\Sigma_{\mathcal{C}}^+)$, $C_0 = D$ and $C_n = B_{S_{m_0}}$. Now, we set $\underline{p} := \Psi((B_{[S_{m_0},S_{P(j)})}C_{[0,n)})^\infty)$. Then $\underline{p}$ is a periodic point with period $l := S_{P(j)} - S_{m_0} + n$. Since $A_{R_j-1} \to B_{S_{P(j)}}$, we have $a_{(R_{j-1},R_j)} = b_{[0,S_{P(j)})}$, which implies (I).

(Case D) $j \in \mathcal{A}_2^{(3)}$**.** Since $P(j)+1 \in \mathcal{B}_2$, we can find $u \leq P(j)$ such that $B_{S_{P(j)+1}-1} \to B_u$. We set $\underline{p} := \Psi((B_{(S_{P(j)},S_{P(j)+1})}B_{[u,S_{P(j)}]})^\infty)$. Then $\underline{p}$ is a periodic point with period $l := S_{P(j)+1} - u$. Since $A_{R_j-1} \to B_{S_{P(j)}}$ and $B_{S_{P(j)+1}-1} \to B_u$, we have $a_{(R_{j-1},R_j)} = b_{[0,S_{P(j)})} = b_{[0,u)}b_{[u,S_{P(j)})} = b_{(S_{P(j)},S_{P(j)+1})}b_{[u,S_{P(j)}]}$. This implies (I).

(Case E) $j \in \mathcal{A}_3$**.** Take $D \in \mathcal{S}(A_{R_j-1}) \cap \mathcal{D}_0$. By the definition of n_1, we can find an integer $n \leq n_1$ and finite vertices $C_0, \ldots, C_n \in \mathcal{C}$ such that $C_{[0,n]} \in \mathcal{L}(\Sigma_{\mathcal{C}}^+)$, $C_0 = D$ and $C_n = A_{R_{m_0}}$. Hence if we set $\underline{p} := \Psi((A_{[R_{m_0},A_{R_j})}C_{[0,n)})^\infty)$, then (I) holds.

(Case F) $j \in \mathcal{A}_4$**.** In this case, $j, j+1 \in \mathcal{A}_2$, $P(j)+1 \in \mathcal{B}_1$, $j \notin \mathcal{A}_3$ and $P(j) \notin \mathcal{B}_3$ hold by the definition of $\mathcal{A}_4$. We prove the following proposition, which plays a fundamental role to prove this case.

Proposition 3.4. We have

$$S_{P(j)}^\infty \succ \{r_{j+i}-1\}_{i=1}^\infty \quad \text{or} \quad R_j^\infty \succ \{s_{P(j)+i}-1\}_{i=1}^\infty.$$

Here $\succ$ denotes the lexicographical order.

Proof. For the notational simplicity, set $r^{(m)} := \{r_{m+i}-1\}_{i=1}^\infty$ and $s^{(m)} := \{s_{m+i}-1\}_{i=1}^\infty$ for $m \geq 1$. Showing by contradiction, we assume

$$S_{P(j)}^\infty \preceq r^{(j)} \quad \text{and} \quad R_j^\infty \preceq s^{P(j)}. \tag{3.1}$$

For $q \geq R_j+1$ set $\mathcal{E}_{q,1} := \{m : R_j \leq R_m < q\}$ and $\mathcal{E}_{q,2} := \{m : S_{P(j)} \leq R_m < q\}$. Denote also $\underline{x} := f_{j+1}(\underline{a})$ and $\underline{y} := \mathrm{adj}(\underline{x}) = f_{P(j)+1}(\underline{b})$. Note that $f_m(\underline{a}) = x$ implies $m \in \mathcal{A}_2$. Similarly $f_m(\underline{b}) = \underline{y}$ implies $m \in \mathcal{B}_1$. To prove the proposition, it is sufficient to show the following for all $q \geq R_j+1$:

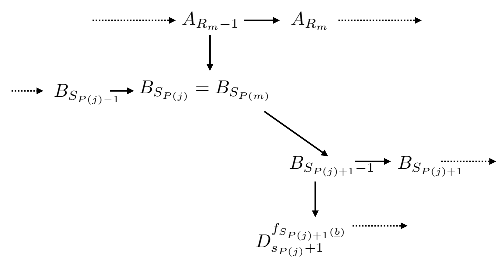

FIGURE 2. Sketch of the Markov diagram in Case 1.

- For any $m \in \mathcal{E}_{q,1}$, we have

$$m \notin \mathcal{A}_3,\ f_{m+1}(\underline{a}) = \underline{x} \text{ and } S_{P(j)}^{\infty} \preceq r^{(m)}. \tag{3.2}$$

- For any $m \in \mathcal{E}_{q,2}$, we have

$$m \notin \mathcal{B}_3,\ f_{m+1}(\underline{b}) = \underline{y} \text{ and } R_j^{\infty} \preceq s^{(m)}. \tag{3.3}$$

Indeed, letting $\mathcal{C}' := \{A_m : m \geq R_j\} \cup \{B_m : m \geq S_{P(j)}\}$, we have $\bigcup_{C \in \mathcal{C}'} \mathcal{S}(C) \cap (\mathcal{C} \setminus \mathcal{C}') = \emptyset$ by (3.2) and (3.3). However, this is a contradiction since $\mathcal{C}$ is irreducible and $A_{n_0} \in \mathcal{C} \setminus \mathcal{C}'$.

We prove (3.2) and (3.3) by induction with regard to $q \geq R_j + 1$. Let $q = R_j + 1$. Since $R_{j+1} \geq R_j + 1$, $j + 1 \notin \mathcal{E}_{R_j+1,1}$. Similarly, $P(j) + 1 \notin \mathcal{E}_{R_j+1,2}$ because $S_{P(j)+1} > s_{P(j)+1} \geq R_j$. Hence we have $\mathcal{E}_{R_j+1,1} = \{j\}$ and $\mathcal{E}_{R_j+1,2} = \{P(j)\}$. Hence (3.2) and (3.3) immediately follows by (3.1).

Let $q > R_j + 1$ and assume the inductive hypothesis for $q - 1$. Take $m \in \mathcal{E}_{q,1} \setminus \mathcal{E}_{q-1,1}$ (if no such m exists, (3.2) automatically holds for all $m \in \mathcal{E}_{q,1}$). We will show that (3.2) holds for m. Since $m \in \mathcal{E}_{q,1}$, we have $R_{m-1} < R_m < q$ and hence $m - 1 \in \mathcal{E}_{q-1,1}$. Therefore, by the inductive hypothesis, we have $m - 1 \notin \mathcal{A}_3$, $f_m(\underline{a}) = \underline{x}$ and $S_{P(j)}^{\infty} \preceq r^{(m-1)}$.

(Case 1) $S_{P(j)} = r_m - 1$ (for the situation of the Markov diagram, see Figure 2). In this case, we have $S_{P(j)} = r_m - 1 = S_{P(m)}$. This implies $A_{R_m-1} \subset B_{S_{P(j)-1}}$ and $\#\mathcal{S}(A_{R_m-1}) \leq \#\mathcal{S}(B_{S_{P(j)-1}}) = 2$. Hence we have $m \notin \mathcal{A}_3$. Moreover, $f_{m+1}(\underline{a}) = \underline{x}$ is followed by $\mathcal{S}(A_{R_m-1}) = \{A_{R_m}, B_{S_{P(j)}}\}$ and $f_{S_{P(j)+1}}(\underline{b}) = \underline{y}$. The last condition $S_{P(j)}^{\infty} \preceq r^{(m)}$ follows from $S_{P(j)} = r_m - 1$ and $S_{P(j)}^{\infty} \preceq r^{(m-1)}$.

(Case 2) $S_{P(j)} < r_m - 1$ (For the situation of the Markov diagram, see Figure 3.4). In this case, we have $S_{P(j)} < r_m - 1 = S_{P(m)} < R_m = q - 1$,

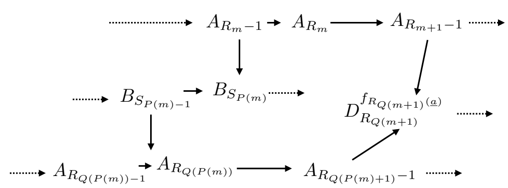

FIGURE 3. Sketch of the Markov diagram in Case 2.

which implies that $P(m)-1,\ P(m) \in \mathcal{E}_{q-1,2}$. Hence by the inductive hypothesis (3.3) for $q-1$, we have $P(m)-1, P(m) \notin \mathcal{B}_3,\ P(m), P(m)+1 \in \mathcal{B}_1$ and $R_j^\infty \preceq s^{(P(m)-1)},\ s^{(P(m))}$. Since $A_{R_m-1} \to B_{S_{P(m)}}$, we have $\#\mathcal{S}(A_{R_m-1}) \le \#\mathcal{S}(B_{S_{P(m)-1}}) = 2$, which implies $m \notin \mathcal{A}_3$. Moreover, $\mathcal{S}(A_{R_m-1}) = \{A_{R_m}, B_{S_{P(m)}}\}$ implies

$$f_{m+1}(\underline{a}) = \mathrm{adj}(f_{S_{P(m)+1}}(\underline{b})) = \mathrm{adj}(\underline{y}) = \underline{x}.$$

Now we consider the last condition $S^\infty_{P(j)} \preceq r^{(m)}$. Since $Q(P(m)) = s_{P(m)} - 1 \ge R_j$ and $Q(P(m)) < s_{P(m)} < S_{P(m)} < R_m = q-1$, (3.2) holds for $Q(P(m))$. If $r^{(Q(P(m)))} = r^{(m)}$, (3.2) yields $S^\infty_{P(j)} \preceq r^{(Q(P(m)))} = r^{(m)}$. Otherwise let $\ell = \inf\{i \ge 1 : r_{Q(P(m))+i} \ne r_{m+i}\}$. It is easy to see $S^\infty_{P(j)} \preceq r^{(m)}$ is yield by the following: For $i = 0, 1, \dots, \ell$, we have

$$A_{R_{m+i}} \subset A_{R_{Q(P(m))+i}} \text{ and } \#\tau(A_{R_{m+i}-1}) = \#\tau(A_{R_{Q(P(m))+i}-1}) = 2 \tag{3.4}$$

In particular, (3.4) implies $r_{Q(P(m))+\ell+1} < r_{m+\ell+1}$ and $r^{(Q(P(m))+1)} \preceq r^{(m)}$. We close our proof of (3.2) by showing (3.4). Since $A_{R_m-1} \to B_{S_{P(m)}}$ and $B_{S_{P(m)}-1} \to A_{R_{Q(P(m))}}$, we have

$$A_{R_m-1} \subset B_{S_{P(m)}-1} \subset A_{R_{Q(P(m))}-1}.$$

Moreover, $f_{m+1}(a) = f_{Q(P(m))+1} = \underline{x}$ implies $A_{R_m}, A_{R_{Q(P(m))}} \subset [x_0]$ and $A_{R_m} \subset A_{R_{Q(P(m))}}$. Since (3.2) holds for $Q(P(m))$, we have $\#\mathcal{S}(A_{R_m-1}) = \#\mathcal{S}(A_{R_{Q(P(m))}-1}) = 2$.

Let $0 < i \le \ell$ and assume (3.4) holds for all $0 \le i' < i$. Since $A_{R_{m+i-1}} \subset A_{R_{Q(P(m))+i-1}}$ and $r_{Q(P(m))+i-1} = r_{m+i-1}$, we have $A_{R_{m+i}-1} \subset A_{R_{Q(P(m))+i}-1}$.

Since $A_{R_{Q(P(m))}} < S_{P((m)} < R_m$ yields

$$\begin{aligned} R_{Q(P(m))+i} &= R_{Q(P(m))} + \sum_{i'=1}^{i} r_{Q(P(m))+i'} \\ &= R_{Q(P(m))} + \sum_{i'=1}^{i} r_{m+i'} \\ &< R_m + R_{m+i} - R_m = R_{m+i}, \end{aligned}$$

we have $\#\mathcal{S}(A_{R_{m+i}-1}) = \#\mathcal{S}(A_{R_{Q(P(m))+i}-1}) = 2$. Combining the definition of ℓ and the (3.4) for ℓ, we have $r_{Q(P(m))+\ell+1} < r_{m+\ell+1}$.

We can prove (3.3) for all $l \in \mathcal{E}_{q,2}$ in a similar manner, which proves the proposition. □

We continue to the proof of Theorem B. By Proposition 3.4, we can divide (Case F) into the following three cases:

(Case F1) $R_j^\infty \succ \{s_{P(j)+i} - 1\}_{i=1}^\infty$.

(Case F2) $S_{P(j)}^\infty \succ \{r_{j+i} - 1\}_{i=1}^\infty$ and $P(j) - 1 \in \mathcal{B}_1$.

(Case F3) $S_{P(j)}^\infty \succ \{r_{j+i} - 1\}_{i=1}^\infty$ and $P(j) - 1 \in \mathcal{B}_2$.

The proofs of **(Case F1)**, **(Case F2)** and **(Case F3)** are similar to those of **(Case 3)**, **(Case 4)** and **(Case 5)** in [16], respectively. Theorem B is proved.

Remark 3.5. In our proof of Theorem B, we improve the method in Hofbauer and Raith's work [16] to apply generalized mod one transformations. For a piecewise monotonic map with two monotonic pieces, its critical set $\mathcal{CR}$ consists of two points which are clearly adjacent. Moreover, its Markov diagram has no vertices at which more than three edges start and a word represented by a path without branching vertices coincides with the word of either of the points in $\mathcal{CR}$. On the other hand, for a generalized mod one transformation, there may exist a vertex at which more than three edges start in its Markov diagram. Moreover it is difficult to check which critical point represent a word defined on a path without branching vertices. Therefore, in the proof of Proposition 3.4 we need to be careful to check that every vertex has two or less edges and a pair of adjacent critical points, $\underline{x} = f_j(\underline{a})$ and $\underline{y} = \mathrm{adj}(\underline{x}) = f_{P(j+1)}(\underline{b})$, appears in each inductive step.

4. Generalizations and further problems

In this section, we describe how Theorem B can be generalized, and further problems related to density of periodic measures. To state our general result precisely, we give necessary definitions. We say that $\underline{x} \in \mathcal{CR}$ is *essential* if there is no integer n so that $\sigma^n(\underline{x})$ is periodic. We denote by $\mathcal{ECR}$ the set of all essential points in $\mathcal{CR}$. We also define an equivalent relation $\sim$ on $\mathcal{ECR}$ by $\underline{x} \sim \underline{y}$ if and only if there is a pair of integers (p, q) so that $\sigma^p(\underline{x}) = \sigma^q(\underline{y})$ holds. Let $\mathcal{ECR}/\sim$ be a quotient space with respect to $(\mathcal{ECR}, \sim)$ and set $\mathcal{E}(T) := \#(\mathcal{ECR}/\sim)$. Roughly speaking, $\mathcal{E}(T)$ is a number of infinite paths of the Markov diagram which can represent any orbit of T. A crucial point of our proof of Theorem B is that orbits of a generalized mod one transformation can be represented by two infinite paths of the Markov diagram corresponding to two essential critical points, and all points in $\mathcal{CR}$ are essential. Hence slightly modifying our proof, we can establish the following general result.

Theorem C. *Let $T\colon [0,1] \to [0,1]$ be a transitive piecewise monotonic map with $h_{\mathrm{top}}(T) > 0$ and assume either of the following holds:*

- $\mathcal{E}(T) \leq 1$.
- $\mathcal{E}(T) = 2$ *and* $\mathcal{CR} = \mathcal{ECR}$.

Then $\mathcal{M}^p_T([0,1])$ is dense in $\mathcal{M}^e_T([0,1])$.

The density of periodic measures for piecewise monotonic maps has been intensively studied by many authors independently of large deviations. We recall that Hofbauer and Raith [16, 23] essentially proposed a problem asking whether $\mathcal{M}^p_T([0,1])$ is dense in $\mathcal{M}^e_T([0,1])$ for any transitive piecewise monotonic maps with positive topological entropy. We note that if a piecewise monotonic map T is continuous, then T satisfies the specification property (see e.g. [2]) and hence density of periodic measures follows from the classical result of Sigmund ([28]). For discontinuous-case, the Hofbauer-Raith problem has a positive answer in the following two cases (see [15, 16]).

- The map T is a mod one transformation, that is, there exists a strictly increasing and continuous function $F\colon [0,1] \to \mathbb{R}$ such that
$$T(x) = \begin{cases} F(x)(\mathrm{mod}\ 1) & (x \neq 1), \\ \lim\limits_{y \to 1-0} (F(y)\ (\mathrm{mod}\ 1)) & (x = 1). \end{cases}$$
- The map T has two intervals of monotonicity, that is, there is a point $0 < c < 1$ such that both $T|_{(0,c)}$ and $T|_{(c,1)}$ are strictly monotonic and continuous.

It is easy to check that Theorem C is applicable to not only generalized mod one transformations, but also known examples listed above.

For directions along which one could try to extend the results of this paper, it is natural to ask whether the second item of the hypothesis in Theorem C, $\mathcal{CR} = \mathcal{ECR}$ can be removed. Moreover, it is also expected that one could try to extend the result of this paper for a piecewise monotonic map T with $3 \leq \mathcal{E}(T) \leq 4$, for example,

- a map such that every (open) interval of monotonicity except the first and the last one is mapped to $(0,1)$,
- a piecewise monotonic map with three monotonic pieces.

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